The Gluon Self-Energy
in
Cavity Quantum Chromodynamics

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A thesis submitted in partial fulfilment of the
requirements for the degree of

Doctor of Philosophy

in the Department of Physics

University of Cape Town

May 1991

Abstract

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A numerical technique to regularize divergent loop diagrams in cavity quantum chromodynamics is discussed, which is closely related to free space dimensional regularization. In this cavity regularization method, the energy shift is expressed as the integral of a divergent spectral function, from which the divergence may be extracted by analogy to the free space expression. It is shown for the case of the self-energy of a gluon in a cavity that no new divergences arise due to the presence of the boundary, provided that the regularization can be achieved in such a way that no subtractions are necessary. In order to avoid such subtractions, the so-called method of separation is developed, in which the spectral forms in the cavity are separated in such a way that the divergences of the various terms cancel exactly. This method is in close analogy to the free space regularization method of separation where tadpole contributions are separated off from the rest of the momentum integrals. The technique is used to evaluate the self-energy of a gluon in a cavity, which turns out to be positive for both the quark loop and the gauge loops. The positive value obtained offers a possible explanation for the absence of gluonic exotic states.

Contents

1 Introduction **1**

2 Regularization of Divergent Loop Diagrams **3**
2.1 Dimensional Regularization in Free Space 3
2.2 Analytic Continuation 5
2.3 Transformation into the Cavity 7
2.4 Regularization in the Cavity 11

3 The Gluon Self-Energy in Free Space **13**
3.1 The Quark Loop 14
3.2 The Ghost Loop 15
3.3 The Gluon Loop 16
3.4 The Tadpole Diagram 17

4 The Boundary **19**
4.1 The Propagators and Boundary Conditions 19
4.1.1 The Quark Propagator 20
4.1.2 The Ghost Propagator 20
4.1.3 The Gluon Propagator 21
4.2 The Gluon Self-Energy in Half-Space QCD 22
4.2.1 The Quark Loop 22
4.2.2 The Gluon Loop 24
4.2.3 The Ghost Loop 27
4.2.4 The Gluon Tadpole 27
4.2.5 The Sum of Reflection Terms 28

5 The Gluon Self-Energy in the Cavity **29**
5.1 The Quark Loop 29
5.2 The Ghost Loop 32
5.3 The Gluon Loop 34
5.4 The Gluon Tadpole 39

6 Calculation and Results **41**
6.1 Calculation . . . 41
6.1.1 The Quark Loop . . . 41
6.1.2 The Gauge Loops . . . 44
6.2 Results . . . 47
6.2.1 The Quark Loop . . . 47
6.2.2 The Gauge Loops . . . 49
6.2.3 Conclusion . . . 51
6.2.4 Acknowledgements . . . 52

A QCD in a Spherical Cavity **53**
A.1 The Cavity Modes . . . 54
A.1.1 The Quark Cavity Modes . . . 54
A.1.2 The Gluon and Ghost Cavity Modes . . . 55
A.2 The Propagators . . . 58
A.2.1 The Quark Propagator . . . 58
A.2.2 The Ghost Propagator . . . 58
A.2.3 The Gluon Propagator . . . 59
A.3 The Vertex Functions . . . 60
A.3.1 The Quark-Gluon Vertex . . . 60
A.3.2 The Ghost-Gluon Vertex . . . 61
A.3.3 The Three-Gluon Vertex . . . 62
A.3.4 The Four-Gluon Vertex . . . 62

B Conventions and Integrals **64**
B.1 Feynman Integrals . . . 64
B.2 Gaussian Integrals . . . 68
B.3 Wick Rotations and Euclidean Space . . . 69
B.4 Conventions . . . 70

C Sum Rules **71**
C.1 The Quark Loop Sum Rule . . . 71
C.2 The Ghost Loop Sum Rule . . . 72
C.3 The Gluon Loop Sum Rules . . . 73

D The Feynman Rules **76**

Chapter 1

Introduction

In the recent past, perturbative QCD in a spherical cavity [1] has been extensively used to predict mass spectra and other characteristics of hadrons [2]. Most of these applications, however, have been limited to the level of tree [3], [4] or box [5] diagrams. Attempts to extend these calculations to divergent loop diagrams in cavity QCD [6]-[12] have met with considerable difficulties, in spite of the fact that QCD in a cavity is presumably a renormalizable relativistic quantum field theory [4]. However, a powerful new technique has recently been introduced which allows the regularization of divergent Feynman diagrams in a cavity [13]. This method, which we will refer to as the cavity regularization technique, entails mimicking the dimensional regularization procedure conventionally used to calculate loop diagrams in free space, without actually having to perform the cavity calculation in arbitrary space-time dimensions.

In dimensional regularization, the finite part of a divergent integral is obtained by subtracting from the total expression, expressed in arbitrary dimensions, its divergent contribution. After this subtraction procedure, the result may be taken in four dimensions. This convenient feature is exploited in the cavity regularization: The divergent quantity is expressed as a spectral form in four dimensions, which contains a non-integrable singularity. Its singular behaviour is exactly as in free space; the cavity spectral form may thus be regularized by subtracting from it the free space singular part.

This method has been employed successfully to calculate the quark self-energy and the "photon" self-energy in scalar QED [13], as well as the electromagnetic corrections to the quark-gluon vertex [14]. However, up to now there has been no attempt to calculate the gluon self-energy in a spherical cavity, even though it is an important ingredient for calculating the properties of glue balls [15] and gluonic exotic states [16]. There are several reasons for this shortcoming: firstly, the calculation obviously promises to be very complicated since the non-Abelian character of QCD involves a multitude of vertices. Secondly, the calculation of the gluon self-energy in the cavity presents new problems which were not present in the quark self-energy. There, the singular part of the loop integral was contained entirely in the portion containing free or unreflected cavity propagators, as has been shown in [6]. In the case of the gluon self-energy, it was not clear, up to now, if there were any additional singularities due to the cavity surface, which would spoil the renormalizability of cavity QCD.

It is argued in this thesis that the divergent contributions to the gluon self-energy due to the boundary cancel out, making a calculation of this quantity possible. The calculation, however, is not straightforward, since one has to *arrange* the cavity expression in such a way as to make the boundary contribution vanish. For this process we rely on an observation made in the free space regularization of the gluon self-energy: the regularization is achieved by expressing the divergent quantity as a function (the so-called spectral form) of a parametric variable z. The result is then the integral of this spectral form over the variable z and is expressed in terms of gamma functions. In this way, the singularity is isolated in the form of a pole of the gamma function. The observation now is that, through the definition of the analytic continuation of the gamma function to the physical region, different spectral forms give rise to the same singularity structure. That means that one can separate the spectral form into several terms in such a way as to ensure that an analytic continuation for the overall expression is not necessary. Such an arrangement ensures that one does not have to subtract anything from the spectral form. In this way, the cancellation of the boundary divergences is maintained, which would not be the case if something were subtracted from the cavity spectral form.

The program in this thesis is as follows: in chapter 2, we discuss the regularization procedure to be employed in the cavity regularization. We also establish the correspondence between the free space and cavity calculations and indicate how a comparison between the two must be made. In chapter 3, we review the free space formalism, deriving the gluon self-energy in the dimensional regularization framework. The effect of the boundary and the reflection behaviour of the propagators at a surface is the subject of chapter 4. Here, we also discuss the free space contribution to the gluon self-energy due to the boundary and argue that its divergent contribution vanishes. This chapter is rather technical and is not necessary for understanding the discussions in subsequent chapters. For easier reading, it may therefore be skipped. The derivation of the corresponding cavity expressions for the gluon self-energy is given in chapter 5. Finally, chapter 6 deals with the methods of calculation and shows the results.

Chapter 2

Regularization of Divergent Loop Diagrams

In order to evaluate ultraviolet divergent loop diagrams, it is necessary to employ a suitable regularization procedure which will make it possible to extract from these diagrams their physically meaningful finite contribution. There exist a whole variety of regularization procedures, the most widely used of which are probably the Pauli-Villars and dimensional regularization methods. In the Pauli-Villars regularization method, the divergent integral is regularized by effectively introducing a cut-off in the momentum integration, which improves the divergent behaviour of the propagator. The regularization method, which we will rely on in this thesis, is the dimensional regularization method, in which the dimension of space-time coordinates D is treated as a continuous variable. It is then noted that the divergence may be isolated by going to an integer value of D.

2.1 Dimensional Regularization in Free Space

In this section, we use the example of the quark loop contribution towards the gluon self-energy to demonstrate the method of dimensional regularization in free space. This pedagogical example will guide us in performing the regularization of the corresponding diagram in the cavity.

In free space, the amplitude for the quark loop diagram is given by the integral

$$i\Pi_Q^{\mu\nu} = -2g^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{2\ell^\mu \ell^\nu + 2\ell^\mu k^\nu - \ell \cdot (\ell + k) g^{\mu\nu}}{[\ell^2 + i0][(\ell + k)^2 + i0]} \tag{2.1}$$

For simplicity, we are putting the quark mass equal to zero here. This integral is regularized in the following manner: after a rotation to Euclidean space, $\ell^0 \to i\ell^0$ (see appendix B.3), the denominator is elevated into an exponential factor using the integral

$$\frac{1}{p^2} = \int_0^\infty dz e^{-p^2 z} \tag{2.2}$$

Generalizing the four-dimensional space-time to $D \equiv 4 - 2\varepsilon$ dimensions, the integral (2.1) becomes

$$
\begin{aligned}
i\Pi_Q^{\mu\nu} &= -2g^2 i \int \frac{d^D\ell}{(2\pi)^D} \int_0^\infty dt_1 \int_0^\infty dt_2\, e^{-\ell^2 t_1} e^{-(\ell+k)^2 t_2} \\
&\quad \times \left(2\ell^\mu \ell^\nu + 2\ell^\mu k^\nu - \ell \cdot (\ell + k)\delta^{\mu\nu} \right) \qquad (2.3)
\end{aligned}
$$

When generalizing to D-dimensional space-time, the coupling constant g should be multiplied by a parameter μ^ε which has the dimension of a mass in order to keep it dimensionless in this arbitrary-dimensional space-time. However, since ultimately all calculations are performed in four dimensions, we shall henceforth neglect this parameter.

Performing a change of variables $t_1 = zt$ and $t_2 = z(1-t)$, and subsequently shifting the momentum integration according to $\ell \to \ell - k(1-t)$ leads to the expression

$$
\begin{aligned}
i\Pi_Q^{\mu\nu} &= -2g^2 i \int \frac{d^D\ell}{(2\pi)^D} \int_0^\infty z dz \int_0^1 dt\, e^{-\ell^2 z - k^2 z t(1-t)} \\
&\quad \times \left(2\ell^\mu \ell^\nu + 2k^\mu k^\nu t(t-1) - \left[\ell^2 + k^2 t(t-1)\right] \delta^{\mu\nu} \right) \qquad (2.4)
\end{aligned}
$$

We have neglected factors which are odd in the momentum variable here since they integrate to zero. The momentum integral is obviously a Gaussian (see appendix B.2) which may be performed immediately. The remaining integral over z yields a gamma function, in which the divergence is isolated in the form of a pole of the gamma function. The relevant integrals are listed in appendix B.1 for convenience.

At this point, before we continue with the calculation, note that one may use the identity

$$
\ell \cdot k = \tfrac{1}{2}(\ell + k)^2 - \tfrac{1}{2}\ell^2 - \tfrac{1}{2}k^2 \qquad (2.5)
$$

to obtain from the original integral (2.1) the expression

$$
i\Pi_Q^{\mu\nu} = -2g^2 \int \frac{d^D\ell}{(2\pi)^D} \left\{ \frac{2\ell^\mu \ell^\nu + 2\ell^\mu k^\nu + \frac{1}{2}k^2 g^{\mu\nu}}{[\ell^2 + i0][(\ell+k)^2 + i0]} - \frac{\frac{1}{2}\ell^2 + \frac{1}{2}(\ell+k)^2}{[\ell^2 + i0][(\ell+k)^2 + i0]} g^{\mu\nu} \right\} \qquad (2.6)
$$

The second term in (2.6) may be evaluated as it stands, or one can cancel common factors of ℓ^2 or $(\ell + k)^2$ and afterwards compute the resulting integral, which is now a sum of tadpole integrals. As has been pointed out by Capper and Leibbrandt [17], the integral

$$
\int \frac{d^D\ell}{(2\pi)^D} \frac{(\ell+k)^2}{\ell^2(\ell+k)^2} = \int \frac{d^D\ell}{(2\pi)^D} \frac{\ell^2}{\ell^2(\ell+k)^2} + \int \frac{d^D\ell}{(2\pi)^D} \frac{k^2}{\ell^2(\ell+k)^2} + \int \frac{d^D\ell}{(2\pi)^D} \frac{2\ell \cdot k}{\ell^2(\ell+k)^2} \qquad (2.7)
$$

is ill-defined because the various terms in the sum on the right-hand side of (2.7) have different regions of analyticity. However, when naïvely calculated with the help of the generalized Feynman integrals from appendix B.1, without worrying about analyticity, the total integral reduces to zero. On the other hand, if we try to evaluate the tadpole integral due to a propagator for massless particles,

$$T = \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{(\ell^2 + i0)} \tag{2.8}$$

we find that our general procedure of evaluating Feynman integrals does not work here. Indeed, rotating to Euclidean space and elevating the denominator, one obtains an ill-defined integral

$$\begin{aligned} T &= -i \int \frac{d^D\ell}{(2\pi)^D} \int_0^\infty dz e^{-\ell^2 z} \\ &= -i \int_0^\infty \left(\frac{1}{4\pi z}\right)^{D/2} dz \end{aligned} \tag{2.9}$$

Capper and Leibbrandt [17] have given a prescription such that this zero-mass tadpole integral may consistently be defined to be zero in the framework of dimensional regularization. It amounts to replacing the mass by a parameter $f(D)$ depending on the space-time dimension D. After the integration, this parameter is set to zero for $D = 4$

$$\begin{aligned} T &= -i \int \frac{d^D\ell}{(2\pi)^D} \int_0^\infty e^{-\ell^2 z - f(D) z} dz \\ &= -i \int_0^\infty \left(\frac{1}{4\pi z}\right)^{D/2} e^{-f(D)z} dz \\ &= \frac{-i}{(4\pi)^{D/2}} \frac{\Gamma(-1+\varepsilon)}{(f(D))^{-1+\varepsilon}} \\ &\equiv 0 \end{aligned} \tag{2.10}$$

This discussion shows that in free space, we may use the tadpole form of the Feynman integral, in other words we may readily cancel any common factors of ℓ^2 from the integral expression without affecting the final result. This is so because the tadpole integral just gives zero, regardless of whether one chooses to cancel factors of momentum squared or not. Therefore it is immaterial what the form of the z-integral is in free space. However, near a boundary or in the cavity, the tadpole integral does contribute to the overall gluon self-energy; the treatment of tadpole contributions and cancellations of this kind must be carefully re-evaluated.

2.2 Analytic Continuation

Let us now proceed to evaluate the Gaussian integral in (2.4). The resulting z-integral, or so-called spectral form, has a different structure depending on whether or not we chose to cancel common factors of momentum squared in the numerator and denominator. Evaluating the integral as it stands and recalling that $\delta_{\mu\nu}\delta^{\mu\nu} = D$, we obtain the spectral form

$$\begin{aligned}
i\Pi_Q^{\mu\nu} &\equiv i\int_0^\infty dz\, \Pi_Q^{\mu\nu}(z) \\
&= -2g^2 \frac{i}{(4\pi)^{D/2}} \int_0^1 dt \int_0^\infty dz z^{-1+\varepsilon} e^{-k^2 zt(1-t)} \\
&\quad \times \left\{ 2\left(k^\mu k^\nu t(t-1) + \frac{\delta^{\mu\nu}}{2z}\right) - \delta^{\mu\nu}\left(k^2 t(t-1) + \frac{D}{2z}\right)\right\} \quad (2.11)
\end{aligned}$$

On the other hand, on cancellation of common factors of ℓ^2 or $(\ell+k)^2$ in the second term of (2.6), we obtain instead

$$\begin{aligned}
i\Pi_Q^{\mu\nu} &= -2g^2 \frac{i}{(4\pi)^{D/2}} \int_0^1 dt \int_0^\infty dz z^{-1+\varepsilon} \quad (2.12) \\
&\quad \times \left\{ e^{-k^2 zt(1-t)} \left[2\left(k^\mu k^\nu t(t-1) + \frac{\delta^{\mu\nu}}{2z}\right) + \tfrac{1}{2}k^2\delta^{\mu\nu}\right] - \frac{e^{-f(D)z}\delta^{\mu\nu}}{z}\right\}
\end{aligned}$$

These spectral forms may now be evaluated to yield a combination of gamma functions. Notice that some of the terms in the integrals give rise to a gamma function of a negative argument, namely $\Gamma(-1+\varepsilon)$. These terms have to be properly defined by analytic continuation of the gamma function to the physical region via the well-known relation

$$n\Gamma(n) \equiv \Gamma(n+1) \quad (2.13)$$

With this in mind, and remembering the definition of the auxiliary parameter $f(D)$, both spectral forms give rise to the same final result

$$i\Pi_Q^{\mu\nu} = 2g^2 \frac{i}{(4\pi)^2}\frac{1}{3}(k^\mu k^\nu - k^2 g^{\mu\nu})\left\{\frac{1}{\varepsilon} - \gamma + \frac{5}{3} - \ln\left(\frac{-k^2}{4\pi}\right)\right\} \quad (2.14)$$

This result is already rotated back to Minkowski space.

Alternatively, the analytic continuation may be achieved by first defining the spectral forms and evaluating the integral afterwards. A convenient way of doing this is via the relation [18]

$$\Gamma(-w) \equiv \int_0^\infty dz\, \frac{e^{-z} - \sum_{k=0}^n (-1)^k \frac{z^k}{k!}}{z^{w+1}} \qquad [n = E(\mathrm{Re}\, w)] \quad (2.15)$$

In other words, in order to arrive at the final result, one has to subtract an analytic continuation factor $\mathcal{C}^{\mu\nu}(z)$ from the spectral form

$$i\Pi_Q^{\mu\nu} = \int_0^\infty dz\, \left(i\Pi_Q^{\mu\nu}(z) - \mathcal{C}_Q^{\mu\nu}(z)\right) \quad (2.16)$$

This subtraction factor has a different form depending on whether tadpole-terms have been separated off or not. Accordingly, the analytic continuation factor for the unseparated spectral form (2.11) is (still in Euclidean space)

$$\mathcal{C}_Q^{\mu\nu}(z) = -2g^2 \frac{i}{(4\pi)^2}\frac{\delta^{\mu\nu}}{z^2}\left(1 - \frac{D}{2}\right) \quad (2.17)$$

whereas for the separated form (2.12), the analytic continuation factors for the two separate terms cancel exactly. This means that if we are able to separate the z-form in this fortuitous fashion, there is no need for an analytic continuation. This property will prove to be very convenient when we attempt the corresponding calculation in the cavity.

In (2.14), the divergence is now isolated in the form of the pole of the gamma function. The finite contribution of this diagram may be obtained by subtracting from it the singular part $\mathcal{S}^{\mu\nu}$

$$\begin{aligned}
\mathcal{S}_Q^{\mu\nu} &\equiv \int_0^\infty dz\, \mathcal{S}_Q^{\mu\nu}(z) \\
&= 2g^2 \frac{i}{(4\pi)^{D/2}} \frac{1}{3}(k^\mu k^\nu - k^2 g^{\mu\nu}) \int_0^\infty dz\, z^{-1+\varepsilon} e^{-z} \\
&= 2g^2 \frac{i}{(4\pi)^2} \frac{1}{3}(k^\mu k^\nu - k^2 g^{\mu\nu}) \left(\frac{1}{\varepsilon} - \gamma + \ln(4\pi)\right)
\end{aligned} \tag{2.18}$$

Here we have expressed the singular part as a spectral form, which is the convenient mode for comparison with other quantities.

Finally, the finite part is given by

$$\begin{aligned}
i\Pi^{\mu\nu}_{Q,\text{finite}} &\equiv \int_0^\infty dz\, \left(i\Pi_Q^{\mu\nu}(z) - \mathcal{S}_Q^{\mu\nu}(z)\right) \\
&= 2g^2 \frac{i}{(4\pi)^2} \frac{1}{3}(k^\mu k^\nu - k^2 g^{\mu\nu}) \left(\frac{5}{3} - \ln(-k^2)\right)
\end{aligned} \tag{2.19}$$

The fact that we chose, as the singular subtraction factor, not only the divergent piece $1/\varepsilon$ but also the constant factors γ and $\ln(4\pi)$, reflects a specific choice of the renormalization prescription.

2.3 Transformation into the Cavity

In this section, we discuss briefly how the cavity diagrams may be obtained from the free space expressions. There are various methods by which this transformation may be achieved.

Firstly, recall that the expression for a loop diagram, as given for example by (2.1), is the Feynman amplitude for the loop section of the diagram only, i.e. the external legs have been amputated for the purpose of the calculation. In order to obtain a quantity which one may calculate numerically, it is advisable to eliminate the tensor structure by restoring the external legs to the diagram, in this way obtaining a scalar quantity. Thus the recipe for a transformation from free space into the cavity is just to restore the external legs, replace all the wave functions by the corresponding cavity modes and finally integrate over the volume of the cavity. This method has the advantage that one can make use of the coordinate space Feynman rules to arrive at the cavity expression.

In our example, the quark loop in free space is given in terms of the coordinate-space Feynman rules as

$$i\Pi^{\mu\nu}_{Qaa'}(x,y) = -\left(-ig\gamma^{\mu}_{\beta\alpha}\frac{\lambda^{a}_{cb}}{2}\right) iS_{\alpha\alpha'}(x,y)\left(-ig\gamma^{\nu}_{\alpha'\beta'}\frac{\lambda^{a'}_{bc}}{2}\right) iS_{\beta'\beta}(y,x) \tag{2.20}$$

According to our recipe, the corresponding quantity in the cavity is given by

$$\Pi_Q(\Sigma,q,\Sigma',q') = \int d^4x \int d^4y\, A^{\mu *}_{\Sigma}(q,x)\,\Pi_{Q,\mu\nu}(x,y)\,A^{\nu}_{\Sigma'}(q',y) \tag{2.21}$$

where Σ and Σ' stand for the polarizations and q and q' for the quantum numbers of the external gluon legs, with $q \equiv \{\omega, m\}$. Into this expression, we may now substitute the propagator and cavity modes, which are derived in appendices A.1 and A.2

$$\begin{aligned} iS_{\alpha\beta}(x,y) &= i\sum_{p}\frac{\psi_\alpha(p,x)\overline{\psi}_\beta(p,y)}{\omega-\epsilon_p \pm i0} && (2.22)\\ \psi(p,x) &= u_n(\vec{x})e^{-i\omega t} && (2.23)\\ A^{\mu}_{\Sigma}(q,x) &= \frac{1}{\sqrt{2\Omega^{\Sigma}_{m}}}a^{\mu}_{\Sigma m}(\vec{x})e^{-i\omega t} && (2.24)\end{aligned}$$

The sum over the quark cavity modes labelled by $p \equiv \{\omega, n\}$ consists of a sum over the cavity quantum numbers $n \equiv \{\nu,\kappa,\mu\}$ and an integral over the continuous energy variable ω

$$\sum_p \equiv \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\sum_n \tag{2.25}$$

The trace of the colour matrices gives rise to the colour factor $T = 1/2$. Expressed in terms of the cavity modes, the quark loop contribution reduces to

$$\begin{aligned}\Pi_Q(\Sigma,q,\Sigma',q') &= -i\,g^2T\sum_{p_1p_2}\frac{1}{2\Omega^{\Sigma}_{m}} \qquad (2.26)\\ &\times\int_{-\infty}^{\infty}dt_x e^{i(\omega-\omega_1+\omega_2)t_x}\int d\vec{x}\left[\overline{u}_{p_2}(\vec{x})i\gamma_\mu u_{p_1}(\vec{x})a^{\mu *}_{\Sigma m}(\vec{x})\right]\int_{-\infty}^{\infty}\frac{d\omega_1}{2\pi}\frac{1}{(\omega_1-\epsilon_{p_1}\pm i0)}\\ &\times\int_{-\infty}^{\infty}dt_y e^{-i(\omega'-\omega_1+\omega_2)t_y}\int d\vec{y}\left[\overline{u}_{p_1}(\vec{y})i\gamma_\nu u_{p_2}(\vec{y})a^{\nu}_{\Sigma' m'}(\vec{y})\right]\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}\frac{1}{(\omega_2-\epsilon_{p_2}\pm i0)}\end{aligned}$$

This expression may be reduced further by performing the time integrations which give rise to delta functions and inserting the definition of the quark-gluon vertex functions discussed in appendix A.3. We thus arrive at

$$\begin{aligned}\Pi_Q(\Sigma,q,\Sigma',q') &= -i\frac{g^2T\pi}{\Omega^{\Sigma}_{m}}\sum_{p_1p_2}\tilde{Q}^{\Sigma m}_{p_2p_1}Q^{\Sigma' m'}_{p_1p_2}\delta(\omega,\omega') \qquad (2.27)\\ &\times\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}\frac{1}{(\omega+\omega_2-\epsilon_{p_1}\pm i0)}\frac{1}{(\omega_2-\epsilon_{p_2}\pm i0)}\end{aligned}$$

This expression is the cavity equivalent of $(-i)$ times the Feynman amplitude. Recall that the perturbative energy shift is given by $(-1)^n i$ times the Feynman amplitude, where n denotes the order in the perturbation expansion. To arrive at the energy shift, which is really the quantity of interest, the above expression has to be multiplied by (-1).

Note that in (2.27) there is still a continuous delta function in the energy parameters of the external legs. Since of course the external wave functions have well defined energy, we replace $2\pi\delta(\omega,\omega')$ by a Kronecker delta $\delta_{\omega,\omega'}$.

A further point worth mentioning is that in the cavity expression, the angular momentum algebra restricts the quantum numbers in such a way that the external legs must have equal quantum numbers. Furthermore, in an on-shell calculation, the energies of the external legs must also be the same, so that one is essentially restricted to an energy shift due to a loop where the in- and outgoing legs have the same quantum numbers and the same polarization. From now on, we shall therefore always set $q = q'$ and $\Sigma = \Sigma'$ from the outset.

As an alternative to the transformation method outlined, the energy shift for the diagram under investigation may be calculated directly. In order to do this, we use the symmetric form of the Gell-Mann and Low theorem as described by Sucher [19]. In this formulation, the energy shift is given by

$$\Delta E = \lim_{\varepsilon\to 0} \frac{i}{2}\varepsilon g \frac{\partial}{\partial g} \ln\langle U(\infty,-\infty)\rangle \tag{2.28}$$

where the time evolution operator is given by Dyson's expansion

$$U(t,t_0) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} dt_1 \ldots \int_{t_0}^{t} dt_n\, T\left(\hat{H}^{\varepsilon}_{int}(t_1)\ldots \hat{H}^{\varepsilon}_{int}(t_n)\right) \tag{2.29}$$

Here, the subscript ε on the interaction Hamiltonian represents the usual adiabatic switching on of the interaction, and the hat on top of the operators indicates the fact that we must perform the calculation in the Dirac picture.

We can now use the QCD interaction Hamiltonian, which is given by (A.2), to evaluate the second-order energy shift. For all except the four-gluon interaction terms in the interaction Hamiltonian, the second-order energy shift arises out of the second term in the perturbative expansion (2.29).

In the case of the quark loop diagram, we obtain, after a Wick decomposition, the energy shift

$$\Delta E = -\frac{i}{2}\lim_{\varepsilon\to 0}\varepsilon g^2 \int d^4x \int d^4y\, e^{-\varepsilon(|t_x|+|t_y|)} \left\langle N\left[\left(\overline{\psi}\frac{\boldsymbol{\lambda}}{2}\cdot \not{A}\psi\right)_x \left(\overline{\psi}\frac{\boldsymbol{\lambda}}{2}\cdot \not{A}\psi\right)_y\right]\right\rangle \tag{2.30}$$

The possible contraction leading to the quark loop diagram has been indicated, and the subscripts x and y are a reminder of the space-time coordinates at which the wave functions are to be taken. This expression leads to two terms, one corresponding to the creation of the gluon at y and the subsequent annihilation of a gluon at x, and one corresponding to the opposite situation. Because of the symmetry in the coordinates x and y, these two terms are identical. We can now proceed as before, inserting the expressions for the cavity modes and propagators in the cavity (2.22)–(2.24) and making sure to perform the time integration before taking the limit $\varepsilon \to 0$. The calculation is more tedious than the one encountered in the naïve application of the Feynman rules, but more rigorous. The result is of course the same so that after some algebra one ends up with

$$\Delta E = i\frac{g^2 T}{2\Omega_m^\Sigma}\sum_{p_1 p_2}\tilde{Q}^{\Sigma m}_{p_2 p_1} Q^{\Sigma m}_{p_1 p_2}\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}\frac{1}{(\omega+\omega_2-\epsilon_{p_1}\pm i0)}\frac{1}{(\omega_2-\epsilon_{p_2}\pm i0)}\delta_{\omega,\omega'} \quad (2.31)$$

which is, just as expected, of the opposite sign to the Feynman amplitude (2.27).

Finally, we need the analogue of the four-momentum vector in free space, in order to arrive at the cavity analogue of the free space singular factors. It is desirable to transform the transverse expression

$$k^\mu k^\nu - k^2 g^{\mu\nu}$$

into the cavity. According to our recipe, we must multiply it with the corresponding external legs and integrate over the space-time coordinates

$$\int d^4x \int d^4y \; A^\mu_\Sigma(q,x)(k^\mu k^\nu - k^2 g^{\mu\nu})A^{\nu *}_{\Sigma'}(q',y)$$

Let us define the vector q^Σ in polarization space as follows

$$\begin{aligned} q^\Sigma &\equiv (q^0, q^{\mathcal{L}}, q^{\mathcal{M}}, q^{\mathcal{E}}) \\ &= (\omega, \Omega, 0, 0) \\ q_\Sigma &= (\omega, -\Omega, 0, 0) \end{aligned} \quad (2.32)$$

where ω is the continuous energy parameter, Ω stands for the eigenenergy of the cavity mode, and the metric in polarization space is defined in the usual way

$$g^{\Sigma\Sigma'} = \begin{cases} 1 : \Sigma = \Sigma' = 0 \\ -1 : \Sigma = \Sigma' = \mathcal{L}, \mathcal{M}, \mathcal{E} \\ 0 : \text{otherwise} \end{cases} \quad (2.33)$$

As required, this definition gives rise to the virtuality

$$q^2 = \omega^2 - \Omega^2 \quad (2.34)$$

Then it may be verified that the expression

$$q^\Sigma q^{\Sigma'} - q^2 g^{\Sigma\Sigma'}$$

is exactly the required cavity analogue of the transverse free space expression. Quite generally, then, we may regard the polarization vector q^Σ as the cavity analogue of the free space momentum vector k^μ, as may be verified explicitly in each case by the transformation procedure outlined here.

2.4 Regularization in the Cavity

We now wish to apply the free space regularization techniques to the cavity. To this end, we note that in going from free space to the cavity, the three-momentum integral reduces to an infinite sum over cavity modes, but the energy integral remains, as in the free space form. Moreover, in the large-momentum or small-z limit, the sum over cavity modes reduces to the free space integral as required. Thus the cavity loop integral (2.27) has a structure similar to that in free space, (2.1). This means that we still retain an energy denominator which corresponds to the momentum-squared denominator in free space. This is the factor which we wish to elevate into an exponential, thereby obtaining for the cavity sum a spectral form of the same structure as the one in free space (2.4).

In this process, we are guided by the expectation that the singular behaviour of the cavity expression must be the same as in free space. For that to be true, one first needs to show that the boundary of the cavity does not introduce any new singularities on top of the ones encountered in the free integral. This task is performed in chapter 3, where it is shown that this is indeed the case.

Recall that we found two different expressions for the spectral form of the loop diagram in free space: firstly, equation (2.11), in which the integration was performed without any prior cancellations in the momentum integral and which therefore requires an analytic continuation to be performed in order to define the expression for its region of analyticity. We refer to this as the unseparated spectral form. Secondly, (2.12) was obtained after cancelling factors of momentum squared in the integral, in other words the tadpole contribution to the integral has been separated off. This separated spectral form does not need any analytic continuation to be performed in order to arrive at the result.

In carrying over this procedure to the cavity, we note that the singular behaviour in the cavity should be the same as in free space, so one expects the same to be true for the analytic continuation procedure, which is really just a subtraction of the highest order divergence. However, in subtracting an analytic continuation factor such as (2.17) or a singular factor such as (2.18), one may subtract additional constant factors since the definition of factors like the Euler constant γ or logarithmic terms may not be the same in the cavity, as they do not actually constitute part of the singular behaviour. It is thus preferable to find a scheme in which these subtractions are not necessary. This is provided by the separated spectral form: here, the analytic continuation is automatic. Furthermore, note that the remaining singular factor in the cavity (which corresponds to the free space singular factor (2.18)) is proportional to the factor

$$q^{\Sigma} q^{\Sigma'} - q^2 g^{\Sigma\Sigma'}$$

This factor is zero for on-shell gluons with magnetic and electric polarizations, which are exactly the polarizations we are interested in. Thus, if we are able to formulate the cavity spectral form in an analogous way to the separated free space spectral form (2.12), no subtraction whatsoever is necessary to achieve the regularization of the cavity diagram. This makes the result independent of the renormalization prescription.

Recall the cavity form of the quark loop diagram

$$\Delta E_Q = i\frac{g^2 T}{2\Omega_m^\Sigma}\sum_{p_1 p_2}\tilde{Q}^{\Sigma m}_{p_2 p_1} Q^{\Sigma m}_{p_1 p_2}\delta_{\omega,\omega'}\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}\frac{1}{(\omega+\omega_2-\epsilon_{p_1})}\frac{1}{(\omega_2-\epsilon_{p_2})} \tag{2.35}$$

In this equation, the Feynman prescription for the poles is implicit. Bringing the energy integral into a form equivalent to the free space form,

$$\begin{aligned} i\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}E(\omega_2) &\equiv i\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}\frac{1}{(\omega+\omega_2-\epsilon_{p_1})}\frac{1}{(\omega_2-\epsilon_{p_2})} \\ &= i\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}\frac{(\omega_2+\epsilon_{p_2})}{\left[\omega_2^2-\epsilon_{p_2}^2\right]}\frac{(\omega_2+\omega+\epsilon_{p_1})}{\left[(\omega+\omega_2)^2-\epsilon_{p_1}^2\right]} \end{aligned} \tag{2.36}$$

we may separate this expression in a way completely analogous to the free space separation in (2.6)

$$\begin{aligned} i\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}E(\omega_2) &= i\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}\frac{(\omega_2+\omega+\epsilon_{p_1})(-\omega_2-\omega+\epsilon_{p_1}+2\omega_2+\omega-\epsilon_{p_1}+\epsilon_{p_2})}{\left[\omega_2^2-\epsilon_{p_2}^2\right]\left[(\omega+\omega_2)^2-\epsilon_{p_1}^2\right]} \\ &= i\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}\left(\frac{-1}{\left[\omega_2^2-\epsilon_{p_2}^2\right]}+\frac{(\omega_2+\omega+\epsilon_{p_1})(2\omega_2+\omega+\epsilon_{p_2}-\epsilon_{p_1})}{\left[\omega_2^2-\epsilon_{p_2}^2\right]\left[(\omega+\omega_2)^2-\epsilon_{p_1}^2\right]}\right) \end{aligned} \tag{2.37}$$

As in free space, common factors of momentum squared $q^2 = \omega_2^2 - \epsilon_{p_2}^2$ have been cancelled between the numerator and the denominator. Note the minus sign between the two terms in (2.37): this corresponds to the minus sign in the free space separation (2.6), so that the separation is made in an entirely analogous fashion to the one in free space.

Of course, all calculations in the cavity are performed in $D = 4$ dimensions. Since we have seen that one may formulate the cavity loop diagrams in such a way that a subtraction of singular factors is not necessary, there is also no need to worry about the limiting procedure of letting $\varepsilon \to 0$. We have thus effectively interchanged the procedures of integrating over z and taking the limit $D \to 4$: in free space, this limit is taken as the last step in the calculation, whereas in the cavity, we start off in $D = 4$ dimensions right away.

Finally, the separated result (2.37) may be transformed into a spectral form using the usual elevation of the denominators into an exponential factor. This expresses the cavity diagram in terms of an integral over z, which contains a sum over cavity modes. The sum over cavity modes is infinite, but may be terminated at some cut-off energy. If the sum is performed first, leaving the integral over z as the last step in the calculation, the exponential factor serves to damp out the terms containing large energies, thus making the error introduced by the energy cut-off small. The details of this calculation are presented in chapter 5.

Chapter 3

The Gluon Self-Energy in Free Space

The QCD vacuum polarization, or gluon self-energy, can be evaluated in free space using the Feynman gauge, where the gluon propagator takes the simple form

$$iD^{ab}_{\mu\nu} = -i\frac{g_{\mu\nu}}{p^2 + i0}\delta_{ab} \tag{3.1}$$

In second order perturbation theory, there are four contributing diagrams: the

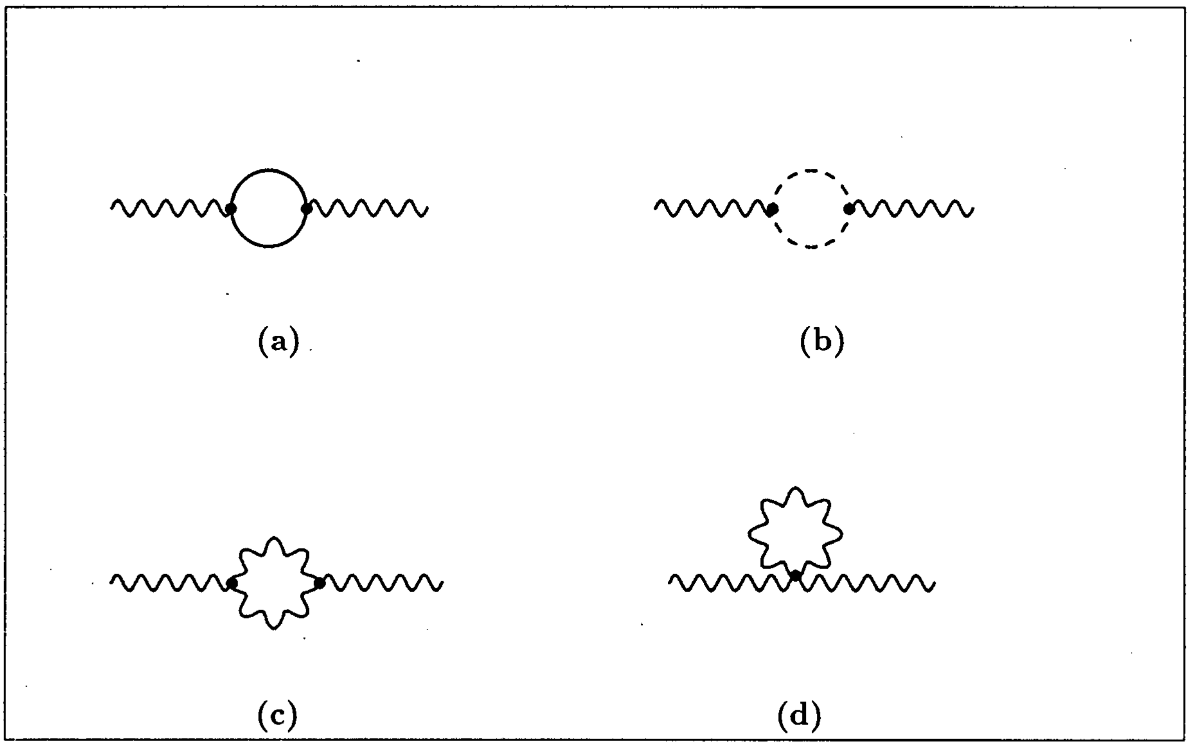


Figure 3.1: *Feynman diagrams contributing to the gluon self-energy*

quark loop $\Pi^{\mu\nu}_Q$, where a gluon momentarily splits into a quark-antiquark pair (fig.

3.1a), the Faddeev-Popov ghost loop $\Pi^{\mu\nu}_{FP}$, in which the gluon changes for a moment into a pair of ghosts (fig. 3.1b), the gluon loop $\Pi^{\mu\nu}_{G}$, which describes the process where the gluon emits and re-absorbs another gluon (fig. 3.1c), and finally the gluon tadpole diagram $\Pi^{\mu\nu}_{T}$ (fig. 3.1d).

We consider each of these diagrams separately.

3.1 The Quark Loop

The quark loop diagram, figure (3.1a), is the only diagram involving quark masses and thus it must be transverse on its own. Using the Feynman rules and assuming that the quarks are massless, the Feynman amplitude for the quark loop is given by

$$i\Pi^{\mu\nu}_{Q,aa'} = -\int \frac{d^D\ell}{(2\pi)^D}\left(-ig\gamma^\mu \frac{\lambda^a_{cb}}{2}\right)\left(i\frac{\delta_{bb'}}{\not\ell + \not k \pm i0}\right)\left(-ig\gamma^\nu \frac{\lambda^{a'}_{b'c'}}{2}\right)\left(i\frac{\delta_{cc'}}{\not\ell \pm i0}\right) \tag{3.2}$$

From the Lorentz structure it is clear that the above expression represents a trace. Evaluating the trace of the colour matrices gives a colour factor T

$$\mathrm{Tr}\left(\frac{\lambda^a}{2}\frac{\lambda^b}{2}\right) = T\,\delta_{ab} = \frac{\delta_{ab}}{2} \tag{3.3}$$

This is the case if only one quark flavour contributes to the vacuum polarization. Of course, if n_f quark flavours are involved, then one gets instead

$$T = \frac{n_f}{2} \tag{3.4}$$

We are therefore led to the expression

$$i\Pi^{\mu\nu}_{Q,aa'} = -Tg^2\delta_{aa'}\int \frac{d^D\ell}{(2\pi)^D}\frac{\mathrm{Tr}(\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta)\ell_\alpha(\ell+k)_\beta}{[\ell^2+i0][(\ell+k)^2+i0]} \tag{3.5}$$

The colour indices on the gluon self-energy represent a colour delta function in the external gluon legs. Therefore one may omit these indices in future without loss of clarity. As discussed in the previous chapter, the auxiliary parameter μ needed to render the coupling constant dimensionless in D dimensions is omitted since in the end we shall require only results in $D = 4$ dimensions.

Using the trace relation

$$\mathrm{Tr}(\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta) = 4(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta} + g^{\mu\beta}g^{\alpha\nu}) \tag{3.6}$$

the expression for the quark loop becomes

$$i\Pi^{\mu\nu}_{Q} = -4Tg^2\int \frac{d^D\ell}{(2\pi)^D}\frac{2\ell^\mu\ell^\nu + 2\ell^\mu k^\nu - \ell^2 g^{\mu\nu} - \ell\cdot k g^{\mu\nu}}{[\ell^2+i0][(\ell+k)^2+i0]} \tag{3.7}$$

Using dimensional regularization, as discussed in chapter 2, leads to the result

$$i\Pi_Q^{\mu\nu} = 4Tg^2 \frac{i}{(4\pi)^2} \frac{1}{3} (k^\mu k^\nu - k^2 g^{\mu\nu}) \left(\frac{1}{\varepsilon} - \gamma + \frac{5}{3} - \ln\left(\frac{-k^2}{4\pi}\right)\right) \tag{3.8}$$

This result is in Minkowski space and satisfies the transversality condition

$$k_\mu \Pi_Q^{\mu\nu} = \Pi_Q^{\mu\nu} k_\nu = 0 \tag{3.9}$$

as required by gauge invariance.

As discussed in the previous chapter, the singular part of (3.8) is given by the spectral form

$$\mathcal{S}_Q^{\mu\nu}(z) = 4Tg^2 \frac{i}{(4\pi)^{D/2}} \frac{1}{3} (k^\mu k^\nu - k^2 g^{\mu\nu}) \frac{e^{-z}}{z^{1-\varepsilon}} \tag{3.10}$$

Finally, the analytic continuation for the unseparated form (3.7) may be expressed as a spectral form and written in Minkowski space as

$$\mathcal{C}_Q^{\mu\nu}(z) = 4Tg^2 \frac{i}{(4\pi)^2} \frac{g^{\mu\nu}}{z^2} \left(1 - \frac{D}{2}\right) \tag{3.11}$$

As we have seen, the analytic continuation factor vanishes for the separated form.

3.2 The Ghost Loop

The Faddeev Popov ghost loop, figure (3.1b), is given in terms of the Feynman rules as

$$i\Pi_{FP,aa'}^{\mu\nu} = -\int \frac{d^D\ell}{(2\pi)^D} (-g f_{dca}(\ell+k)^\mu) \frac{i\delta_{cc'}}{[(\ell+k)^2 + i0]} (-g f_{c'd'a'}\ell^\nu) \frac{i\delta_{dd'}}{[\ell^2 + i0]} \tag{3.12}$$

The sum over the structure constants gives rise to a colour factor

$$f_{acd} f_{a'dc} \equiv C\delta_{aa'} = -3\delta_{aa'} \tag{3.13}$$

so that one ends up with

$$i\Pi_{FP,aa'}^{\mu\nu} = Cg^2 \delta_{aa'} \int \frac{d^D\ell}{(2\pi)^D} \frac{(\ell+k)^\mu \ell^\nu}{[\ell^2 + i0][(\ell+k)^2 + i0]} \tag{3.14}$$

Note that the ghost loop does not contain any tadpole terms. After dimensional regularization and a rotation back to Minkowski space, one obtains the result

$$\begin{aligned} i\Pi_{FP}^{\mu\nu} &= Cg^2 \frac{i}{(4\pi)^2} \left\{ -\frac{1}{6} k^\mu k^\nu \left(\frac{1}{\varepsilon} - \gamma + \frac{5}{3} - \ln\left(\frac{-k^2}{4\pi}\right)\right) \right. \\ &\qquad \left. - \frac{1}{12} k^2 g^{\mu\nu} \left(\frac{1}{\varepsilon} - \gamma + \frac{8}{3} - \ln\left(\frac{-k^2}{4\pi}\right)\right)\right\} \end{aligned} \tag{3.15}$$

As expected, this result is not transverse on its own. The singular part for this loop is obviously given by

$$\mathcal{S}^{\mu\nu}_{FP}(z) = Cg^2 \frac{i}{(4\pi)^{D/2}} \left(-\frac{1}{6} k^\mu k^\nu - \frac{1}{12} k^2 g^{\mu\nu} \right) \frac{e^{-z}}{z^{1-\varepsilon}} \tag{3.16}$$

Finally, there is no possible separation of this loop, so we obtain the analytic continuation factor (given in Minkowski space)

$$\mathcal{C}^{\mu\nu}_{FP}(z) = -Cg^2 \frac{i}{(4\pi)^2} \frac{g^{\mu\nu}}{z^2} \left(\frac{1}{2} \right) \tag{3.17}$$

3.3 The Gluon Loop

Next, we evaluate the gluon loop diagram. The Feynman rules in momentum space give rise to the integral expression

$$i\Pi^{\mu\nu}_{G,aa'} = \frac{1}{2} \int \frac{d^D\ell}{(2\pi)^D} (-g f_{dac} V^{\tau\sigma\mu}) \left(\frac{-i\delta_{cc'} g_{\sigma\sigma'}}{(\ell+k)^2 + i0} \right) (-g f_{c'a'd'} V^{\tau'\sigma'\nu}) \left(\frac{-i\delta_{dd'} g_{\tau\tau'}}{\ell^2 + i0} \right) \tag{3.18}$$

where we have abbreviated

$$V^{\tau\sigma\mu} \equiv (\ell + 2k)^\tau g^{\sigma\mu} - (2\ell + k)^\mu g^{\tau\sigma} - (k - \ell)^\sigma g^{\mu\tau} \tag{3.19}$$

The factor $\frac{1}{2}$ is the usual symmetry factor. The integral expression for the gluon loop thus becomes

$$\begin{aligned} i\Pi^{\mu\nu}_G &= -\frac{C}{2} g^2 \delta_{aa'} \int \frac{d^D\ell}{(2\pi)^D} \times \qquad (3.20) \\ &\frac{(4D-6)\ell^\mu\ell^\nu + (2D-3)(\ell^\mu k^\nu + k^\mu \ell^\nu) + (D-6)k^\mu k^\nu + \{2\ell\cdot(\ell+k) + 5k^2\} g^{\mu\nu}}{[\ell^2 + i0][(\ell+k)^2 + i0]} \end{aligned}$$

As before, the trivial colour factor will be omitted henceforth. This expression may again be separated into two terms, in one of which all common factors of momentum squared may be cancelled

$$\begin{aligned} i\Pi^{\mu\nu}_G &= -\frac{C}{2} g^2 \int \frac{d^D\ell}{(2\pi)^D} \left\{ \frac{\ell^2 + (\ell+k)^2}{[\ell^2 + i0][(\ell+k)^2 + i0]} g^{\mu\nu} \right. \qquad (3.21) \\ &\left. + \frac{(4D-6)\ell^\mu\ell^\nu + (2D-3)(\ell^\mu k^\nu + k^\mu\ell^\nu) + (D-6)k^\mu k^\nu + 4k^2 g^{\mu\nu}}{[\ell^2 + i0][(\ell+k)^2 + i0]} \right\} \end{aligned}$$

Of course, both the unseparated and the separated version lead to the same result

$$\begin{aligned} i\Pi^{\mu\nu}_G &= -\frac{C}{2} g^2 \frac{i}{(4\pi)^2} \left\{ -\frac{11}{3} k^\mu k^\nu \left(\frac{1}{\varepsilon} - \gamma + \frac{67}{33} - \ln\left(\frac{-k^2}{4\pi} \right) \right) \right. \\ &\left. + \frac{19}{6} k^2 g^{\mu\nu} \left(\frac{1}{\varepsilon} - \gamma + \frac{116}{57} - \ln\left(\frac{-k^2}{4\pi} \right) \right) \right\} \qquad (3.22) \end{aligned}$$

Similarly, the singular part naturally does not depend on the separation and is given in its spectral representation by

$$\mathcal{S}_G^{\mu\nu}(z) = -\frac{C}{3}g^2\frac{i}{(4\pi)^{D/2}}\left(\tfrac{1}{2}(D-15)k^\mu k^\nu + \left(-\frac{D}{2}+\frac{27}{4}\right)k^2 g^{\mu\nu}\right)\frac{e^{-z}}{z^{1-\varepsilon}} \tag{3.23}$$

For the analytic continuation factor there are again two possibilities. The unseparated integral (3.20) gives rise to the factor

$$\mathcal{C}_G^{\mu\nu}(z) = \frac{C}{2}g^2\frac{i}{(4\pi)^2}\frac{g^{\mu\nu}}{z^2}(3D-3) \tag{3.24}$$

whereas the separated form (3.21) leads to

$$\mathcal{C}_G^{\mu\nu}(z) = \frac{C}{2}g^2\frac{i}{(4\pi)^2}\frac{g^{\mu\nu}}{z^2}(2D-1) \tag{3.25}$$

3.4 The Tadpole Diagram

The gluon tadpole diagram, figure (3.1d), is obtained from the four-gluon vertex. The Feynman rules give

$$\begin{aligned} i\Pi_{T,aa'}^{\mu\nu} &= -\frac{i}{2}g^2\int\frac{d^D\ell}{(2\pi)^D}\left(\frac{-ig_{\sigma\sigma'}}{\ell^2+i0}\right)\Big[f_{cce}f_{a'ae}(g^{\sigma'\nu}g^{\sigma\mu}-g^{\sigma'\mu}g^{\sigma\nu})+ \\ &\quad f_{ca'e}f_{cea}(g^{\sigma'\mu}g^{\sigma\nu}-g^{\sigma\sigma'}g^{\mu\nu})+f_{cae}f_{ca'e}(g^{\sigma\sigma'}g^{\mu\nu}-g^{\sigma'\nu}g^{\sigma\mu})\Big] \end{aligned} \tag{3.26}$$

Again, the symmetry factor $\frac{1}{2}$ has been duly included. This diagram is just a tadpole integral

$$i\Pi_T^{\mu\nu} = \frac{Cg^2}{2}g^{\mu\nu}\,2(D-1)\int\frac{d^D\ell}{(2\pi)^D}\frac{1}{\ell^2+i0} \tag{3.27}$$

As has been discussed in chapter 2, this diagram is usually defined to be zero in dimensional regularization. Because this diagram does not contribute anything in free space, it obviously also does not give rise to a singular factor either. However, even though the tadpole integrates to zero, the analytic continuation factor is not zero as may be verified immediately by inspection of the spectral form for the tadpole as given for example in (2.10). It is given by

$$\mathcal{C}_T^{\mu\nu}(z) = -\frac{C}{2}g^2\frac{i}{(4\pi)^2}\frac{g^{\mu\nu}}{z^2}2(D-1) \tag{3.28}$$

On comparing the analytic continuation factors for the three gauge loop diagrams, one notices that when the gluon loop is expressed in the separated form, these three continuation factors add up to zero. It is therefore important to include the tadpole graph in the cavity calculation, as we wish to exploit exactly this important feature:

if the total analytic continuation factor vanishes, it means that no subtractions are necessary for the cavity calculation.

Finally, we must add the three diagrams containing gauge particles in the loop together in order to obtain a transverse or gauge invariant expression. As discussed, the tadpole diagram does not contribute to the free space result, so the result in free space is given as the sum of the gluon and ghost loops

$$i\Pi^{\mu\nu}_{G} + i\Pi^{\mu\nu}_{FP} = \frac{C}{2}g^2\frac{i}{(4\pi)^2}(k^\mu k^\nu - k^2 g^{\mu\nu})\frac{10}{3}\left(\frac{1}{\varepsilon} - \gamma + \frac{31}{15} - \ln\left(\frac{-k^2}{4\pi}\right)\right) \tag{3.29}$$

The total vacuum polarization due to the quark and gauge loops is given by

$$\Pi^{\mu\nu} = \frac{g^2}{16\pi^2}\left(k^\mu k^\nu - k^2 g^{\mu\nu}\right)\left(\left(\frac{2n_f}{3} - 5\right)\left(\frac{1}{\varepsilon} - \gamma - \ln\frac{-k^2}{4\pi}\right) + \frac{10n_f}{9} - \frac{31}{3}\right) \tag{3.30}$$

Chapter 4

The Boundary

As mentioned, a *conditio sine qua non* for the success of a calculation of the vacuum polarization in a cavity is that no new divergences arise due to the boundary, as this would spoil the renormalizability of the cavity field theory. We can think of the cavity vacuum polarization as the sum of two distinct contributions: a part due to the free or unreflected propagators and one where the propagators undergo any number of reflections from the cavity surface. To investigate whether new singularities occur in the presence of the surface, we have to take a look at the reflection contribution to the vacuum polarization. However, the calculation of propagators multiply reflected from a spherical surface is a forbidding task. The singularities in Feynman diagrams arise as short-distance or large-momentum singularities, i.e. the loop integral becomes divergent as we approach one of the endpoints of the propagator, or as the parametric variable z approaches zero, or as the loop momentum approaches infinity. This means that we are interested in the behaviour of the reflection contribution at short distances from the boundary, as this is where the singularities arise. At very short distances from a spherical surface this curved surface looks like a plane, so we treat the reflections of the propagators as reflections from an infinite flat surface. This will give us the correct leading order (i.e. most singular) behaviour for the corresponding reflection contributions in a spherical cavity. If we find that the one-reflection contributions are finite for short distances from the boundary, then they must be finite everywhere and any contribution due to more than one reflection must also be finite. Therefore it is sufficient in that case to discuss the one-reflection contribution due to a reflection from a flat surface. Furthermore, the flat surface allows us to use the method of images, which renders the calculation accessible.

4.1 The Propagators and Boundary Conditions

In this section we take a look at how the MIT boundary conditions (see (A.4)–(A.6)) affect the propagators reflected from a flat cavity surface. The boundary here is a $D-1$-dimensional plane positioned at $x_1 = 0$.

4.1.1 The Quark Propagator

The quark propagator obeying the boundary condition

$$(i\hat{r}\cdot\vec{\gamma}+1)S(x,y)|_{x_1=0}=0 \tag{4.1}$$

where $\hat{r}$ denotes the unit vector normal to the boundary surface and pointing outwards, can be divided into a free, or unreflected, and a reflected part as follows

$$S(x,y)=S^0(x,y)+\tilde{S}(x,y) \tag{4.2}$$

As discussed, the propagator can only undergo one reflection, since we are working in an infinite half-space here. In this case, the method of images allows us to express the full propagator as a sum of direct and image terms

$$S(x,y)=S^0(x,y)+\varrho S^0(x_\perp,y) \tag{4.3}$$

where the image position is given by

$$x_\perp=(x_0,-x_1,x_2,x_3) \tag{4.4}$$

and the reflection parameter ϱ must be determined from the boundary condition. Using the representation (4.3), the boundary conditions become

$$(i\hat{r}\cdot\vec{\gamma}+1)\int\frac{d^D\ell}{(2\pi)^D}\left\{\frac{(i\partial\!\!\!/_x+m)e^{-i(x-y)\cdot\ell}+\varrho(i\partial\!\!\!/_{x_\perp}+m)e^{-i(x_\perp-y)\cdot\ell}}{\ell^2-m^2}\right\}\Bigg|_{x_1=0}=0 \tag{4.5}$$

Taking the trace on both sides we arrive at

$$\begin{aligned}
0 &= \int\frac{d^D\ell}{(2\pi)^D}\left\{\frac{(-\frac{\partial}{\partial x}+m)e^{-i(x-y)\cdot\ell}+\varrho(-\frac{\partial}{\partial x_\perp}+m)e^{-i(x_\perp-y)\cdot\ell}}{\ell^2-m^2}\right\}\Bigg|_{x_1=0}\\
&= \int\frac{d^D\ell}{(2\pi)^D}\,\frac{(-\frac{\partial}{\partial x}+m)e^{-i(x-y)\cdot\ell}(1+\varrho)}{\ell^2-m^2}\Bigg|_{x_1=0}
\end{aligned} \tag{4.6}$$

This leads to the value $\varrho=-1$ for the reflection parameter. In other words, the full propagator may be written as

$$S(x,y)=S^0(x,y)-S^0(x_\perp,y) \tag{4.7}$$

4.1.2 The Ghost Propagator

Next, the ghost propagator $\Delta(x,y)$ obeying the boundary conditions

$$\hat{r}\cdot\vec{\nabla}\Delta(x,y)|_{x_1=0}=0 \tag{4.8}$$

can be expressed again in terms of direct and image terms as follows

$$\frac{\partial}{\partial x}\left(\Delta^0(x,y)+\eta\Delta^0(x_\perp,y)\right)\Big|_{x_1=0}=0 \tag{4.9}$$

Using $\left(\frac{\partial}{\partial x_1}\right)_\perp = -\frac{\partial}{\partial x_1}$, this yields the defining equation

$$\begin{aligned} 0 &= \left\{\frac{\partial}{\partial x}\Delta^0(x,y) - \eta\frac{\partial}{\partial x_\perp}\Delta^0(x_\perp,y)\right\}\Bigg|_{x_1=0} \\ &= (1-\eta)\frac{\partial}{\partial x}\Delta^0(x,y)\Bigg|_{x_1=0} \end{aligned} \tag{4.10}$$

and thus $\eta = 1$ or

$$\Delta(x,y) = \Delta^0(x,y) + \Delta^0(x_\perp,y) \tag{4.11}$$

4.1.3 The Gluon Propagator

Finally, the gluon wave functions A^μ obey the boundary conditions

$$\begin{aligned} \hat{r}\cdot\vec{\nabla}A^0(\vec{r})|_{x_1=0} &= 0 \\ \hat{r}\cdot\vec{A}(\vec{r})|_{x_1=0} &= 0 \\ \hat{r}\times(\vec{\nabla}\times\vec{A}(\vec{r}))|_{x_1=0} &= 0 \end{aligned} \tag{4.12}$$

which, for a flat boundary, reduce to the simple form

$$\begin{aligned} \partial_{x_1}A^0|_{x_1=0} &= 0 \\ A^1|_{x_1=0} &= 0 \\ \partial_{x_1}A^2|_{x_1=0} &= 0 \\ \partial_{x_1}A^3|_{x_1=0} &= 0 \end{aligned} \tag{4.13}$$

Again expressing the gluon propagator in the suggestive form

$$D_{\mu\nu}(x,y) = D^0_{\mu\nu}(x,y) + \rho D^0_{\mu\nu}(x_\perp,y) \tag{4.14}$$

we can see immediately, by arguments entirely analogous to the ones for the quark and ghost propagators, that for $\mu \neq 1$, one obtains $\rho = +1$ and for $\mu = 1$, one gets instead $\rho = -1$. Hence, the full propagator assumes the form

$$D_{\mu\nu}(x,y) = D^0_{\mu\nu}(x,y) + (-1)^{n_\mu} D^0_{\mu\nu}(x_\perp,y) \tag{4.15}$$

where we have introduced the shorthand

$$n_\mu = \begin{cases} 1 : \mu = 1 \\ 0 : \text{otherwise} \end{cases} \tag{4.16}$$

This completes the discussion of the boundary conditions and their implications for the propagator reflections from the cavity surface.

4.2 The Gluon Self-Energy in Half-Space QCD

We now evaluate individual contributions to the half-space vacuum polarization, i.e. the gluon self-energy containing both direct and reflected propagators. In order to formulate the propagators in terms of image positions, we need the Feynman rules in coordinate space, which are derived in appendix D. One can then proceed to calculate the full vacuum polarization in the presence of the boundary, using the coordinate space Feynman rules and the half-space propagators from section 4.1. In the case of the quark, gluon and ghost loops, the full vacuum polarization will have the following structure: in each case, the loop contains two propagators which each can be either reflected or unreflected. This product of propagators can be written out as

$$\begin{aligned}\Delta(x,y)\Delta(y,x) = \Delta^0(x,y)\Delta^0(y,x) + \Delta^0(x,y)\Delta^0(y_\perp,x) \\ + \Delta^0(x,y_\perp)\Delta^0(y,x) + \Delta^0(x,y_\perp)\Delta^0(y_\perp,x) \quad (4.17)\end{aligned}$$

where $\Delta(x,y)$ here stands for the quark, gluon or ghost propagator. Correspondingly the vacuum polarization is a sum of the terms

$$\Pi_{\mu\nu}(x,y) = \Pi^0_{\mu\nu}(x,y) + \tilde{\Pi}_{\mu\nu}(x,y) + \tilde{\Pi}_{\mu\nu}(x,y_\perp) + \tilde{\tilde{\Pi}}_{\mu\nu}(x,y_\perp) \quad (4.18)$$

$\Pi^0_{\mu\nu}(x,y)$ contains only unreflected propagators, and $\tilde{\tilde{\Pi}}_{\mu\nu}(x,y_\perp)$ contains only reflected propagators and is equivalent to a vacuum polarization due to an image point. Of course, we expect any new singularities due to the boundary to occur in the terms $\tilde{\Pi}_{\mu\nu}(x,y_\perp)$ and $\tilde{\Pi}_{\mu\nu}(x,y)$, which are due to one reflected - and one free propagator. In the following we discuss the four diagrams separately.

4.2.1 The Quark Loop

Using the Feynman rules in coordinate space we arrive at, for the quark loop

$$\begin{aligned} i\Pi^{\mu\nu}_{Q,aa'}(x,y) &= -\left(-ig\gamma^\mu \frac{\lambda^a_{cb}}{2}\right)\int \frac{d^Dq}{(2\pi)^D}\frac{e^{-iq\cdot(y-x)}}{\not{q}\pm i0}\left(-ig\gamma^\nu\frac{\lambda^{a'}_{b'c'}}{2}\right) \\ &\quad \times \int\frac{d^D\ell}{(2\pi)^D}\frac{e^{-i\ell\cdot(x-y)}}{\not{\ell}\pm i0}\delta(q,\ell+k) \quad (4.19)\end{aligned}$$

In the following, we assume the Feynman prescription for the propagators to be implicit. Let us first look at $\tilde{\Pi}^{\mu\nu}_Q(x,y)$, which is due to the propagator combination $S^0(x,y)S^0(y_\perp,x)$ and is given by

$$\begin{aligned}\tilde{\Pi}^{\mu\nu}_{Q,aa'}(x,y) &= 4iTg^2\delta_{aa'}\int\frac{d^Dk}{(2\pi)^D}e^{-ik\cdot(y-x)}\int\frac{d^D\ell}{(2\pi)^D}\frac{e^{-2i\ell_1 y_1}}{\ell^2(\ell+k)^2} \quad (4.20) \\ &\quad\times\left(2\ell^\mu\ell^\nu + 2\ell^\mu k^\nu + \tfrac{1}{2}k^2g^{\mu\nu} - \tfrac{1}{2}\ell^2 g^{\mu\nu} - \tfrac{1}{2}(\ell+k)^2 g^{\mu\nu}\right)\end{aligned}$$

For convenience, we henceforth omit the trivial colour delta function $\delta_{aa'}$. We may now use the Feynman integrals from appendix B.1 to evaluate this expression. In Euclidean space one arrives at

$$\begin{aligned}
\tilde{\Pi}_Q^{\mu\nu}(x,y) \;=\; & -4Tg^2 \frac{1}{(4\pi)^{D/2}} \int \frac{d^Dk}{(2\pi)^D} e^{-ik\cdot(y-x)} \Bigg\{ \int_0^1 dt e^{-2iy_1k_1t} \Big(2(\tfrac{1}{2}g^{\mu\nu} I(-1+\varepsilon) \\
& - \tfrac{1}{4}x^\mu x^\nu I(-2+\varepsilon) - \frac{it}{2}(k^\mu x^\nu + x^\mu k^\nu) I(-1+\varepsilon) + t^2 k^\mu k^\nu I(\varepsilon) \\
& + \frac{i}{2} k^\mu x^\nu I(-1+\varepsilon) - k^\mu k^\nu \, t \, I(\varepsilon) + \tfrac{1}{2}k^2 g^{\mu\nu} I(\varepsilon) \Big) \\
& - \tfrac{1}{2} g^{\mu\nu} I(-1+\varepsilon) - \tfrac{1}{2} g^{\mu\nu} e^{-i2k_1y_1} I(-1+\varepsilon) \Bigg\}
\end{aligned} \tag{4.21}$$

where we have introduced the abbreviation

$$x^\mu \equiv (0, 2y_1, 0, 0) \tag{4.22}$$

We would now like to investigate this expression further to see whether any divergences survive. The function $I(\nu)$, discussed in appendix B.1, has the asymptotic short distance behaviour

$$\begin{aligned}
I(0) &\sim \ln x \\
I(-1) &\sim x^{-2} \\
I(-2) &\sim x^{-4}
\end{aligned} \tag{4.23}$$

The vector potential component $A^1(x)$ obeys the Dirichlet boundary condition. Therefore it must have the asymptotic form

$$A^1(x) \sim x \tag{4.24}$$

Further, we investigate only the behaviour at short distances from the boundary. This means that y_1 is small. The expression for the vacuum polarization in the cavity is obtained from the free space one by a suitable transformation, as discussed in section 2.3. Hence we will eventually end up with a quantity like

$$\Pi(\Sigma, q, \Sigma', q') = \int d^4x \, d^4y \; A^\mu(\Sigma, q, x) \, \Pi_{\mu\nu}(x,y) \; A^{\nu *}(\Sigma', q', y) \tag{4.25}$$

Looking now at the terms in (4.21), it is clear that any term containing a factor x^μ or x^ν picks out the cavity mode obeying the Dirichlet boundary condition, thus making that term regular. Therefore, the only problematic terms are the ones involving $g^{\mu\nu} I(-1+\varepsilon)$. Now, since we are close to the boundary,

$$\begin{aligned}
\int \frac{d^Dk}{(2\pi)^D} e^{-ik\cdot(y-x)} \int_0^1 dt e^{-2ik_1y_1t} &= \int_0^1 dt \, \delta(y - x + 2ty_1) \\
&\simeq \delta(y-x)
\end{aligned} \tag{4.26}$$

One can easily convince oneself that this is so by multiplying the delta function with an arbitrary function, expanding the function for a small argument and then evaluating the integral. Similarly,

$$\int \frac{d^D k}{(2\pi)^D} e^{-ik\cdot(y-x)} e^{-2ik_1 y_1} \simeq \delta(y-x) \tag{4.27}$$

and

$$\int \frac{d^D k}{(2\pi)^D} e^{-ik\cdot(y-x)} = \delta(y-x) \tag{4.28}$$

In view of this result, it is clear that the three divergent terms in (4.21) cancel against each other, thus leaving the remaining expression finite and integrable.

Now consider the term $\tilde{\Pi}_{\mu\nu}(x, y_\perp)$. This is due to the propagator combination $S^0(x, y_\perp)S^0(y, x)$. The calculation goes along the lines of that for $\tilde{\Pi}_{\mu\nu}(x, y)$, but note that in this case we are left with a momentum integral of the form

$$\begin{aligned} \int \frac{d^D q}{(2\pi)^D} e^{-iq\cdot(y_\perp - x)} \int \frac{d^D \ell}{(2\pi)^D} e^{-i\ell\cdot(x-y)} \quad &\rightarrow \quad \int \frac{d^D k}{(2\pi)^D} e^{-ik\cdot(y-x)+2ik_1 y_1 + 2ik_1 y_1 t} \\ &\rightarrow \int_0^1 dt \delta(y - x - 2y_1 - 2y_1 t) = 0 \end{aligned} \tag{4.29}$$

The last step is due to the fact that we are operating in the half-space where the x-coordinate can assume only positive values, thus making it impossible for the argument of the delta function to vanish.

Finally, the term $\tilde{\tilde{\Pi}}_{\mu\nu}(x, y_\perp)$ must be considered. This term is regular as has been discussed by Stoddart [13]. This is what one would expect since this term is like a free space vacuum polarization due to an image point.

4.2.2 The Gluon Loop

The gluon loop is the most difficult to calculate. It is therefore convenient to explore some of the symmetries of the propagator to arrive at the result. This diagram, in coordinate space, is given by

$$\begin{aligned} i\Pi_G^{\mu\nu}(x,y) &= \frac{1}{2} \int \frac{d^D q}{(2\pi)^D} \int \frac{d^D \ell}{(2\pi)^D} \left(-g f_{dac}\right) V^{\tau\sigma\mu} \left(\frac{-i g_{\sigma\sigma'}}{\ell^2}\right) e^{-i\ell\cdot(x_1 - y_1)} \\ &\quad \times \left(-g f_{c'a'd'}\right) V^{\tau'\sigma'\nu} \left(\frac{-i g_{\tau'\tau}}{q^2}\right) e^{-iq\cdot(y_2 - x_2)} \end{aligned} \tag{4.30}$$

where now

$$\begin{aligned} V^{\tau\sigma\mu} &= (-2i\partial_{x_2} - i\partial_{x_1})^\tau g^{\sigma\mu} + (i\partial_{x_2} - i\partial_{x_1})^\mu g^{\sigma\tau} + (2i\partial_{x_1} + i\partial_{x_2})^\sigma g^{\mu\tau} \\ V^{\tau'\sigma'\nu} &= (-2i\partial_{y_1} - i\partial_{y_2})^{\sigma'} g^{\nu\tau'} + (i\partial_{y_1} - i\partial_{y_2})^\nu g^{\sigma'\tau'} + (2i\partial_{y_2} + i\partial_{y_1})^{\tau'} g^{\sigma'\nu} \end{aligned} \tag{4.31}$$

Note that $V^{\tau'\sigma'\nu}$ is obtained from $V^{\tau\sigma\mu}$ by the symmetry operation $x \leftrightarrow y$, $1 \leftrightarrow 2$, and substituting the corresponding indices. Note also that the Feynman rules used here have been expressed in a form which implicitly takes care of momentum conservation. We now explore the symmetry of the propagator

$$\begin{aligned}\partial^\mu_{x_2}\partial^\nu_{y_1} D_{\sigma\sigma'}(x_1,y_1)D_{\tau\tau'}(y_2,x_2) &= \partial^\mu_{x_1} D_{\sigma\sigma'}(x_1,y_1)\partial^\nu_{y_2} D_{\tau\tau'}(y_2,x_2)\\ \partial^\mu_{x_1}\partial^\nu_{y_1} D_{\sigma\sigma'}(x_1,y_1)D_{\tau\tau'}(y_2,x_2) &= D_{\sigma\sigma'}(x_1,y_1)\partial^\mu_{x_2}\partial^\nu_{y_2} D_{\tau\tau'}(y_2,x_2)\end{aligned} \tag{4.32}$$

After some manipulation we obtain for the free space result

$$\begin{aligned} i\Pi^{\mu\nu}_G(x,y) &= -\frac{C}{2}g^2\int\frac{d^Dq}{(2\pi)^D}e^{-iq\cdot(y-x)}\frac{1}{q^2}\int\frac{d^D\ell}{(2\pi)^D}e^{-i\ell\cdot(x-y)}\frac{1}{\ell^2}\Big\{(4D-6)\ell^\mu\ell^\nu\\ &\quad+(4D-16)\ell^\mu k^\nu+(2D-2)k^\mu\ell^\nu+(2D-12)k^\mu k^\nu\\ &\quad+(2p^2+5k^2+2p\cdot k)g^{\mu\nu}\Big\}\end{aligned} \tag{4.33}$$

This looks a little different to the expression (3.20) we obtained before, but, due to the symmetry in the k - and ℓ integrations, the final result will be the same. Now that we are satisfied that the coordinate space Feynman rules are correct, one can derive the boundary terms. For this purpose one uses the form (4.15) for the reflected propagator and obtains for the full propagator

$$\begin{aligned} i\Pi^{\mu\nu}_G(x,y) &= -\frac{C}{2}g^2\int\frac{d^Dq}{(2\pi)^D}\int\frac{d^D\ell}{(2\pi)^D}V^{\tau\sigma\mu}V^{\tau'\sigma'\nu} \qquad (4.34)\\ &\quad\times\Big\{D^0_{\sigma\sigma'}(x,y)D^0_{\tau'\tau}(y,x)+(-)^{n_\sigma}(-)^{n_{\tau'}}D^0_{\sigma\sigma'}(x,y_\perp)D^0_{\tau'\tau}(y_\perp,x)\\ &\quad+(-)^{n_\sigma}D^0_{\sigma\sigma'}(x,y_\perp)D^0_{\tau'\tau}(y,x)+(-)^{n_{\tau'}}D^0_{\sigma\sigma'}(x,y)D^0_{\tau'\tau}(y_\perp,x)\Big\}\end{aligned}$$

The first term is the free space result, the second term is the vacuum polarization due to an image point, and the fourth term is zero because, as we have seen in the case of the quark loop, the argument of the delta function cannot become zero. The third term is $\tilde{\Pi}^{\mu\nu}_G(x,y)$, which we now investigate. After some algebra one obtains

$$\begin{aligned} i\tilde{\Pi}^{\mu\nu}_G(x,y) &= -\frac{C}{2}g^2\int\frac{d^Dk}{(2\pi)^D}e^{-ik\cdot(y-x)}\int\frac{d^D\ell}{(2\pi)^D}e^{-2i\ell_1y_1}\frac{1}{\ell^2(\ell+k)^2}\\ &\Big\{\Big(4g^\alpha_\alpha(-)^{n_\alpha}-3(-)^{n_\mu}-3(-)^{n_\nu}\Big)\ell^\mu\ell^\nu+\Big(4g^\alpha_\alpha(-)^{n_\alpha}-8(-)^{n_\mu}-8(-)^{n_\nu}\Big)\ell^\mu k^\nu\\ &+\Big(2g^\alpha_\alpha(-)^{n_\alpha}-(-)^{n_\mu}-(-)^{n_\nu}\Big)k^\mu\ell^\nu+\Big(2g^\alpha_\alpha(-)^{n_\alpha}-6(-)^{n_\mu}-6(-)^{n_\nu}\Big)k^\mu k^\nu\\ &+\ell_\alpha\ell^\alpha\Big((-)^{n_\nu}+1\Big)g^{\mu\nu}+5k_\alpha k^\alpha(-)^{n_\nu}g^{\mu\nu}+k_\alpha\ell^\alpha\Big(6(-)^{n_\nu}-4\Big)g^{\mu\nu}\Big\}\end{aligned} \tag{4.35}$$

Using the standard Feynman integrals and inserting the relation (2.5), we see that the only problematic terms arise from the integrals

$$\int\frac{d^D\ell}{(2\pi)^D}\frac{\ell^\mu\ell^\nu}{\ell^2(\ell+k)^2},\quad \int\frac{d^D\ell}{(2\pi)^D}\frac{\ell_\alpha\ell^\alpha}{\ell^2(\ell+k)^2}\quad\text{and}\quad\int\frac{d^D\ell}{(2\pi)^D}\frac{(\ell+k)_\alpha(\ell+k)^\alpha}{\ell^2(\ell+k)^2} \tag{4.36}$$

All other terms in (4.35) contribute only regular factors, so we omit them here. Using the standard Feynman integrals of appendix B.1, one arrives at

$$\begin{aligned}\int \frac{d^D\ell}{(2\pi)^D}\frac{\ell^\mu\ell^\nu}{\ell^2(\ell+k)^2}e^{-2i\ell_1 y_1} &= \frac{i}{(4\pi)^{D/2}}\int_0^1 dt\, e^{-2itk_1y_1}\Big\{\tfrac{1}{2}g^{\mu\nu}I(-1+\varepsilon) \\ &\quad -\tfrac{1}{4}x^\mu x^\nu I(-2+\varepsilon)+t^2\, k^\mu k^\nu I(\varepsilon) \\ &\quad -\frac{it}{2}(k^\mu x^\nu + x^\mu k^\nu)I(-1+\varepsilon)\Big\} \qquad (4.37)\end{aligned}$$

$$\begin{aligned}\int \frac{d^D\ell}{(2\pi)^D}\frac{\ell_\alpha\ell^\alpha}{\ell^2(\ell+k)^2}e^{-2i\ell_1 y_1} &= \frac{i}{(4\pi)^{D/2}}\int_0^1 dt\, e^{-2itk_1y_1}\Big\{\tfrac{1}{2}g^\alpha_\alpha I(-1+\varepsilon) \\ &\quad -\tfrac{1}{4}x_\alpha x^\alpha I(-2+\varepsilon) - itk_\alpha x^\alpha I(-1+\varepsilon) \\ &\quad + t^2 k_\alpha k^\alpha I(\varepsilon)\Big\} \qquad (4.38)\end{aligned}$$

$$\int \frac{d^D\ell}{(2\pi)^D}\frac{(\ell+k)_\alpha(\ell+k)^\alpha}{\ell^2(\ell+k)^2}e^{-2i\ell_1 y_1} = e^{2ik_1y_1}\int \frac{d^D\ell}{(2\pi)^D}\frac{\ell_\alpha\ell^\alpha}{\ell^2(\ell+k)^2}\, e^{-2i\ell_1y_1} \qquad (4.39)$$

Now with the definition of the integral $I(\nu)$ given in appendix B.1, it can be shown that

$$\begin{aligned}\frac{1}{2}g^\alpha_\alpha I(-1+\varepsilon)-\frac{1}{4}x_\alpha x^\alpha I(-2+\varepsilon) &= \frac{D}{2}I(-1+\varepsilon)-\frac{1}{4}x^2 I(-2+\varepsilon) \\ &= I(-1+\varepsilon) \qquad (4.40)\end{aligned}$$

With this simplification, one finally ends up with

$$\int \frac{d^D\ell}{(2\pi)^D}\frac{\ell_\alpha\ell^\alpha}{\ell^2(\ell+k)^2}e^{-2i\ell_1 y_1} = e^{-2ik_1y_1}I(-1+\varepsilon) + \text{regular terms} \qquad (4.41)$$

and

$$\int \frac{d^D\ell}{(2\pi)^D}\frac{(\ell+k)_\alpha(\ell+k)^\alpha}{\ell^2(\ell+k)^2}e^{-2i\ell_1 y_1} = I(-1+\varepsilon) + \text{regular terms} \qquad (4.42)$$

So to leading order, we have for the gluon loop

$$\begin{aligned}i\tilde{\Pi}^{\mu\nu}_G(x,y) &= -\frac{C}{2}g^2\frac{i}{(4\pi)^{D/2}}\int\frac{d^Dk}{(2\pi)^D}e^{-ik\cdot(y-x)}\int_0^1 dt\, I(-1+\varepsilon)g^{\mu\nu} \\ &\quad\times\Big\{\tfrac{1}{2}[4g^\alpha_\alpha(-)^{n_\alpha}-6(-)^{n_\nu}]e^{-2itk_1y_1} \\ &\quad + [-2(-)^{n_\nu}+3]e^{-2ik_1y_1}+3(-)^{n_\nu}-2\Big\} \qquad (4.43)\end{aligned}$$

In the limit $y_1 \to 0$, one can ignore the exponentials involving y_1, so each term is just multiplied by a factor $\delta(y-x)$. The final result to leading order is

$$i\tilde{\Pi}^{\mu\nu}_G(x,y) \to -\frac{C}{2}g^2\frac{ig^{\mu\nu}}{(4\pi)^{D/2}}\Big(2(D-2)-2(-)^{n_\nu}+1\Big)\, I(-1+\varepsilon)\,\delta(y,x) \qquad (4.44)$$

4.2.3 The Ghost Loop

The ghost loop in coordinate space is given as follows

$$\begin{aligned} i\Pi^{\mu\nu}_{FP}(x,y) &= -\int \frac{d^Dq}{(2\pi)^D}\int \frac{d^D\ell}{(2\pi)^D}\left(-gf_{cda}i\partial^{\mu}_{x_1}\right)\frac{i\delta_{cc'}}{q^2}e^{-iq\cdot(x_1-y_1)} \\ &\quad \times\left(-gf_{d'c'a'}i\partial^{\nu}_{y_2}\right)\frac{i\delta_{dd'}}{\ell^2}e^{-i\ell\cdot(y_2-x_2)}\delta(q,\ell+k) \end{aligned} \tag{4.45}$$

This gives us the familiar free space result

$$i\Pi^{\mu\nu}_{FP}(x,y) = Cg^2\int \frac{d^Dk}{(2\pi)^D}e^{-ik\cdot(y-x)}\int \frac{d^D\ell}{(2\pi)^D}\frac{\ell^\mu\ell^\nu+\ell^\mu k^\nu}{\ell^2(\ell+k)^2} \tag{4.46}$$

The term $i\tilde{\Pi}^{\mu\nu}_{FP}(x,y)$ is obtained as before from the propagator combination $\Delta^0(x,y)\ \Delta^0(y_\perp,x)$. The reflection contribution is thus recovered from (4.46) by replacing $\int\frac{d^D\ell}{(2\pi)^D}$ by $\int\frac{d^D\ell}{(2\pi)^D}e^{-2i\ell_1y_1}$. The only contribution to the leading order divergent part arises from the integral

$$\int \frac{d^D\ell}{(2\pi)^D}\frac{\ell^\mu\ell^\nu e^{-2i\ell_1y_1}}{\ell^2(\ell+k)^2} \tag{4.47}$$

With the same procedures as before, we thus obtain the leading order contribution to the reflection part of the ghost loop

$$i\tilde{\Pi}^{\mu\nu}_{FP}(x,y) \to \frac{C}{2}g^2\frac{ig^{\mu\nu}}{(4\pi)^{D/2}}I(-1+\varepsilon)\delta(y,x) \tag{4.48}$$

4.2.4 The Gluon Tadpole

The Feynman rules for the four-gluon vertex in coordinate space are the same as in momentum space. The free space result can thus be written down immediately as

$$i\Pi^{\mu\nu}_{T}(x,y) = \frac{C}{2}g^2g^{\mu\nu}2(D-1)\int \frac{d^Dk}{(2\pi)^D}e^{-ik\cdot(y-x)}\int \frac{d^D\ell}{(2\pi)^D}\frac{1}{\ell^2} \tag{4.49}$$

In this expression, we now use the full propagator

$$D_{\mu\nu}(x,y) = D^0_{\mu\nu}(x,y) + (-1)^{n_\mu}D^0_{\mu\nu}(x_\perp,y) \tag{4.50}$$

to obtain the full half-space contribution. The reflection term is easily seen to be

$$i\tilde{\Pi}^{\mu\nu}_{T}(x,y) = \frac{C}{2}g^2g^{\mu\nu}\int \frac{d^Dk}{(2\pi)^D}e^{-ik\cdot(y-x)}\int \frac{d^D\ell}{(2\pi)^D}e^{-2i\ell_1y_1}\frac{1}{\ell^2}2\left(D-2-(-)^{n_\nu}\right) \tag{4.51}$$

This term is singular. It is finally evaluated to yield

$$i\tilde{\Pi}^{\mu\nu}_{T}(x,y) = \frac{C}{2}g^2\frac{ig^{\mu\nu}}{(4\pi)^{D/2}}2\left(D-2-(-)^{n_\nu}\right)I(-1+\varepsilon)\delta(y,x) \tag{4.52}$$

4.2.5 The Sum of Reflection Terms

It has been shown that the reflection contribution due to the quark loop is finite and integrable. Thus it remains to discuss the reflection contribution due to the sum of the three other diagrams. Using the results of sections 4.2.2 to 4.2.4, it can now be checked easily that the sum of the three loops indeed gives us a regular expression: We have in each case written down only the non-integrable terms in the reflection parts of the gluon and ghost loops and the gluon tadpole. The sum of these divergent terms (4.44), (4.48) and (4.52) is zero, and thus we can be satisfied that there are no new divergences arising from the presence of the boundary in the cavity vacuum polarization.

This concludes the discussion of the vacuum polarization reflection terms in a half-space bounded by an infinite flat surface. It has been found that the one-reflection terms in this approximation add up to give a finite and integrable boundary contribution. This treatment is sufficient to prove that the boundary terms due to an arbitrary number of reflections in the spherical cavity are finite. The reasons for this statement are obvious: firstly, one expects the "worst", i.e. most singular, boundary behaviour to occur in the one-reflection term, since the singularities arise as a short-distance phenomenon. The more reflections there are, the greater the path of the total propagator, and thus the less singular it will be. Secondly, one can think of the spherical surface as built up of a sequence of flat surfaces. This validates the treatment of the curved surface as flat.

One point worth mentioning is the following: We have shown that there is no *new* singularity due to the boundary, but only *if nothing is subtracted* from the expressions derived. It is not clear what will happen if we, for example, subtract terms in order to regularize the divergent integrals. In fact, in the case of the gauge loops, we will find that this subtraction spoils the cancellation of boundary divergences, so that the calculation works only when we separate the spectral forms in such a way that no subtractions are necessary.

Chapter 5

The Gluon Self-Energy in the Cavity

In this chapter we derive the cavity expressions for the four loops contributing to the gluon self-energy. Recall that we may either use the Feynman rules to arrive at the desired result, or alternatively evaluate the expression explicitly from the Gell-Mann and Low Theorem. This convenient fact supplies us with a reliable check on the derivation of the cavity expressions. In what follows, we shall display in each case only the method which is the easiest to follow.

5.1 The Quark Loop

Let us evaluate the quark loop diagram using the symmetric form of the Gell-Mann and Low theorem. The relevant term in the interaction Hamiltonian is the quark-gluon interaction, the first term in (A.2). The second order energy shift is given by

$$\Delta E_Q = -\frac{i}{2}g^2 \int d^4x \int d^4y \left\langle T\left[\left(\overline{\psi}\frac{\boldsymbol{\lambda}}{2}\cdot \not{A}\psi\right)_x \left(\overline{\psi}\frac{\boldsymbol{\lambda}}{2}\cdot \not{A}\psi\right)_y\right]\right\rangle \tag{5.1}$$

In this expression we have omitted the adiabatic switching-on factor $e^{-\varepsilon(|t_x|+|t_y|)}$ as well as the limiting procedure $\varepsilon \to 0$. The result obtained by leaving these terms out is the same as the one obtained by including them. This is the case as long as there are no inconsistencies introduced into the integration contour by omitting this adiabatic switching-on factor, as its sole function is to define the proper contour for the integration. In this case it is thus sufficient to use the cavity expressions for the propagators, with the implicit Feynman prescription at the poles.

We note that because of the symmetry in the coordinates x and y,

$$\left\langle T\left[\left(\overline{\psi}\not{A}\psi\right)_x \left(\overline{\psi}\not{A}\psi\right)_y\right]\right\rangle = 2\left\langle N\left[\left(\overline{\psi}_x\psi_y\right)\left(\psi_x\overline{\psi}_y\right)\not{A}_x^{(+)}\not{A}_y^{(-)}\right]\right\rangle \tag{5.2}$$

This means that one can choose the outgoing line at one of the co-ordinates x or y and cancel the symmetry factor $\frac{1}{2}$.

Inserting the propagators and wave functions in terms of cavity modes, as discussed in section (2.1.3), we obtain the expression

$$\Delta E_Q = i\frac{g^2T}{2\Omega^\Sigma_p}\sum_{p_1p_2}\tilde{Q}^{\Sigma p}_{p_2p_1}Q^{\Sigma p}_{p_1p_2}\delta_{\omega,\omega'}\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}\frac{1}{(\omega+\omega_2-\epsilon_1\pm i0)}\frac{1}{(\omega_2-\epsilon_2\pm i0)} \tag{5.3}$$

where the result has been expressed in terms of the quark-gluon vertex functions (appendix A.3.1), and the time integrals have been performed to yield delta functions. The energies ϵ_1 and ϵ_2 denote the eigenenergies of the quark modes p_1 and p_2 over which the sum is performed. As discussed in chapter 2, we must now express the energy integral as a z-integral by elevating the denominators into exponentials. Let us illustrate this procedure for the unseparated energy factor as it appears in (5.3). First, we express the denominator in the usual momentum-squared form necessary to perform the elevation of denominators

$$\begin{aligned} i\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}E(\omega_2) &= i\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}\frac{1}{(\omega_2-\epsilon_2)(\omega_2+\omega-\epsilon_1)} \\ &= i\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}\frac{(\omega_2+\epsilon_2)(\omega_2+\omega+\epsilon_1)}{[\omega_2^2-\epsilon_2^2][(\omega_2+\omega)^2-\epsilon_1^2]} \end{aligned} \tag{5.4}$$

We rotate to Euclidean space, $\omega_2 \to i\omega_2$, $\omega \to i\omega$ and elevate the denominators into exponentials to obtain

$$\begin{aligned} i\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}E(\omega_2) &= -\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}\int_0^\infty dt_1\int_0^\infty dt_2\, e^{\left(-\omega_2^2t_2-\epsilon_2^2t_2-(\omega_2+\omega)^2t_1-\epsilon_1^2t_1\right)} \\ &\quad\times\Big(-\omega_2(\omega_2+\omega)+i\omega_2(\epsilon_1+\epsilon_2)+i\omega\epsilon_2+\epsilon_1\epsilon_2\Big) \end{aligned} \tag{5.5}$$

Now we perform the shift of variables

$$\omega_2 \to \omega_2 - \frac{t_1}{t_1+t_2} \tag{5.6}$$

and furthermore redefine $t_1 = zt$, $t_2 = z(1-t)$ to obtain

$$\begin{aligned} i\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}E(\omega_2) &= -\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}\int_0^1 dt\int_0^\infty dz\, e^{z\left(-\omega_2^2-\epsilon_2^2(1-t)-\epsilon_1^2t-\omega^2t(1-t)\right)} \\ &\quad\times\Big(-(\omega_2-\omega t)(\omega_2+\omega(1-t))+i(\omega_2-\omega t)(\epsilon_1+\epsilon_2)+i\omega\epsilon_2+\epsilon_1\epsilon_2\Big) \end{aligned} \tag{5.7}$$

The Gaussian integral is then easily evaluated with the help of the integrals in appendix B.2. We obtain, after rotating back to Minkowski space,

$$\begin{aligned} i\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}E(\omega_2) &= -\int_0^1 dt\int_0^\infty dz\sqrt{\frac{z}{4\pi}}\, e^{z\left(\omega^2t(1-t)-\epsilon_2^2(1-t)-\epsilon_1^2t\right)} \\ &\quad\times\Big(-\frac{1}{2z}+\omega^2t(t-1)+\omega\epsilon_2-\omega t(\epsilon_1+\epsilon_2)+\epsilon_1\epsilon_2\Big) \end{aligned} \tag{5.8}$$

We have discussed in section (2.4) that in order to regularize the cavity expression, it is desirable to express it in a form where tadpole terms are separated from the rest of the integral. Thus for the separated form of the quark loop energy integral, which we have given in (2.37), we obtain instead the spectral form

$$i\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}E(\omega_2) \equiv i\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}\left(T(\omega_2)+R(\omega_2)\right) \tag{5.9}$$

where the tadpole contribution is given by

$$\begin{aligned} i\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}T(\omega_2) &\equiv i\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}\frac{-1}{w_2^2-\epsilon_2^2} \\ &= -\int_0^{\infty}dz\sqrt{\frac{1}{4\pi z}}\,e^{-\epsilon_2^2 z} \end{aligned} \tag{5.10}$$

and the remainder of the energy integral becomes

$$\begin{aligned} i\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}R(\omega_2) &= -\int_0^1 dt\int_0^{\infty}dz\sqrt{\frac{z}{4\pi}}\,e^{z\left(\omega^2 t(1-t)-\epsilon_2^2(1-t)-\epsilon_1^2 t\right)} \\ &\times\left\{-\frac{1}{z}+2\omega^2t^2+\Big(-3\omega^2-\omega(\epsilon_1+\epsilon_2)\Big)t+\omega\epsilon_2+\omega^2+\epsilon_1(\epsilon_2-\epsilon_1)\right\} \end{aligned} \tag{5.11}$$

One may evaluate these energy integrals as contour integrals and then perform the sum in (5.3). This gives rise to a quadratically divergent sum, i.e. the sum diverges quadratically with the energy cut-off. This is to be expected from the free space integral form of the vacuum polarization. The spectral form, as it is given for example in (5.11), expresses the energy integral in such a way that terms containing large energies are damped out by the exponential factor, so that the error introduced by neglecting terms corresponding to an energy higher than, say $E_{\max}$, becomes negligible. This shows that the correct cut-off for the sum (5.3) is a cut-off in energy and not in variables like the angular momentum or principal quantum numbers, provided of course we formulate it in terms of a spectral form as described. It is this formulation of the energy integral as a spectral form that actually performs the regularization of the diagram.

The contour integrals of the energy factor may be performed to give a check on the spectral forms, as these are numerically difficult to calculate. The contour integral for the energy denominator in (5.3) is

$$i\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}E(\omega_2) = \left(\frac{\Theta(\epsilon_2)\Theta(-\epsilon_1)}{\omega+\epsilon_2-\epsilon_1}-\frac{\Theta(\epsilon_1)\Theta(-\epsilon_2)}{\omega+\epsilon_2-\epsilon_1}\right) \tag{5.12}$$

where the step function is defined in the usual way

$$\Theta(\epsilon) = \begin{cases} 1 : \epsilon > 0 \\ 0 : \text{otherwise} \end{cases} \tag{5.13}$$

Of course, the contour integral must be correct regardless of whether the separated or the unseparated spectral form is used.

Finally, we have obtained the cavity quark loop diagram as a sum of two spectral forms, the tadpole contribution and a remainder:

$$\Delta E_Q = \frac{g^2}{4\pi}\frac{T}{2\Omega_p^\Sigma}4\pi \sum_{p_1 p_2} \tilde{Q}^{\Sigma p}_{p_2 p_1} Q^{\Sigma p}_{p_1 p_2} \delta_{\omega,\omega'} \left(i \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} \left(T(\omega_2) + R(\omega_2) \right) \right) \tag{5.14}$$

The spectral forms are given in (5.10) and (5.11). In this expression the sum over cavity modes is to be performed first, leaving the integral over z as the last step in the calculation. The sum over the vertex functions

$$4\pi \sum_{p_1 p_2} \tilde{Q}^{\Sigma p}_{p_2 p_1} Q^{\Sigma p}_{p_1 p_2} \tag{5.15}$$

may be checked numerically with the help of a sum rule, in which the completeness relation of the cavity modes is exploited to obtain an independent approximate result for the vertex expression. The sum rules are given in appendix C. The factor $g^2/(4\pi)$ is equal to the strong coupling constant α_s and is not included in the actual calculation. The integral over t may be calculated numerically in terms of error functions and normalized error functions.

Of course, the singular and analytic continuation factors in the cavity are obtained by direct transformation into the cavity from the free space expressions (3.10) and (3.11). The singular factor is given by

$$\mathcal{S}_Q^{\Sigma\Sigma'}(z) = -\frac{\alpha_s}{2\sqrt{\Omega_m^\Sigma \Omega_{p'}^{\Sigma'}}}\frac{T}{3\pi}(q^\Sigma q^{\Sigma'} - q^2 g^{\Sigma\Sigma'})\frac{e^{-z}}{z} \tag{5.16}$$

and the analytic continuation factor for the unseparated form is

$$\mathcal{C}_Q^{\Sigma\Sigma'}(z) = \frac{\alpha_s}{2\sqrt{\Omega_m^\Sigma \Omega_{p'}^{\Sigma'}}}\frac{T}{\pi}\frac{g^{\Sigma\Sigma'}}{z^2} \tag{5.17}$$

The singular factor (5.16) is always zero for on-shell transverse magnetic or electric gluons. That means that once one has gotten rid of the $1/z^2$ divergence by way of a separation or a subtraction, the resulting spectral form should be integrable and yield a finite result.

5.2 The Ghost Loop

Next, we wish to evaluate the ghost loop. We start off with the symmetric form of the Gell-Mann and Low theorem. For the external gluon in a vector polarization, we require the last term from the interaction Hamiltonian (A.2). The second last term is needed only if the external gluon is a scalar, but we do not discuss this case, as the gluon self-energy vanishes in that situation. We thus obtain for the Feynman amplitude

$$\begin{aligned}\Delta E_{FP} &= -\frac{i}{2}\lim_{\varepsilon\to 0}\varepsilon g^2\int d^4x\int d^4y e^{-\varepsilon(|t_x|+|t_y|)}\times \\ &\quad \left\langle T\left[\left(i\partial_k\boldsymbol{\chi}\cdot(\boldsymbol{A}^k\times\boldsymbol{\omega})\right)_x\left(i\partial_l\boldsymbol{\chi}\cdot(\boldsymbol{A}^l\times\boldsymbol{\omega})\right)_y\right]\right\rangle \end{aligned} \tag{5.18}$$

Again, the factors involving ε serve to define the integration contour and may be omitted without affecting the result. For the ghost loop diagram, this leads to the contractions

$$\begin{aligned}\Delta E_{FP} &= -\frac{i}{2}g^2 f_{abc}f_{a'b'c'}\int_{-\infty}^{\infty}dt_x\int_{-\infty}^{\infty}dt_y\int d\vec{x}\int d\vec{y}\times \\ &\quad \left\langle N\left[\left(i\partial_k\,\chi_a\,A_b^k\,\omega_c\right)_x\left(i\partial_l\,\chi_{a'}\,A_{b'}^l\,\omega_{c'}\right)_y\right]\right\rangle \end{aligned} \tag{5.19}$$

Inserting the ghost propagator and cavity modes into (5.19) yields

$$\begin{aligned}\Delta E_{FP} &= \frac{i}{2}g^2 C\sum_{p_1p_2}\int_{-\infty}^{\infty}dt_x\int d\vec{x}\int_{-\infty}^{\infty}dt_y\int d\vec{y}\,\delta_{ac'}\,\delta_{a'c} \\ &\quad\times\int_{-\infty}^{\infty}\frac{d\omega_1}{2\pi}\frac{a_{p_1}^{0*}(\vec{y})i\partial_k a_{p_1}^0(\vec{x})}{\omega_1^2-\Omega_1^2}e^{-i\omega_1(t_x-t_y)}\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}\frac{a_{p_2}^0(\vec{x})i\partial_l a_{p_2}^{0*}(\vec{y})}{\omega_2^2-\Omega_2^2}e^{-i\omega_2(t_x-t_y)} \\ &\quad\times\frac{1}{2\sqrt{\Omega_q^{\Sigma}\Omega_{q'}^{\Sigma'}}}\left\{a_{\Sigma q}^k(\vec{x})a_{\Sigma' q'}^{l*}(\vec{y})e^{-i\omega t_x+i\omega' t_y}+a_{\Sigma' q'}^{k*}(\vec{x})a_{\Sigma q}^l(\vec{y})e^{i\omega' t_x-i\omega t_y}\right\}\end{aligned} \tag{5.20}$$

Here Ω_1 and Ω_2 refer to the eigenenergies of the ghost cavity modes characterized by p_1 and p_2, and (Σ,q) and (Σ',q') refer to the quantum numbers of the external gluons.

The time integration may be performed to yield delta functions in the continuous energy parameters. It turns out that the two expressions obtained from the first and second terms in the last line of (5.21) give the same result, so we may again cancel the symmetry factor $\frac{1}{2}$ and use one of the two expressions.

The derivative acting on the ghost wave function has the effect of returning the wave function of the longitudinal gluon

$$i\vec{\nabla}a_p^0(\vec{x}) = -\Omega_p^0\,\vec{a}_{\mathcal{L}p}(\vec{x}) \tag{5.21}$$

where Ω_p^0 refers to the eigenenergy of the scalar mode with quantum numbers p.

Hence one obtains

$$\begin{aligned}\Delta E_{FP} &= i\frac{g^2C}{2\sqrt{\Omega_q^{\Sigma}\Omega_{q'}^{\Sigma'}}}\sum_{p_1p_2}\Omega_1\Omega_2\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}\frac{1}{[\omega_2^2-\Omega_2^2]}\frac{1}{[(\omega+\omega_2)^2-\Omega_1^2]} \\ &\quad\times\int d\vec{x}\left(\vec{a}_{\mathcal{L}p_1}(\vec{x})\cdot\vec{a}_{\Sigma q}(\vec{x})\,a_{p_2}^0(\vec{x})\right)\int d\vec{y}\left(\vec{a}_{\mathcal{L}p_2}^*(\vec{y})\cdot\vec{a}_{\Sigma' q'}^*(\vec{y})\,a_{p_1}^{0*}(\vec{y})\right)\end{aligned} \tag{5.22}$$

and this may be expressed in terms of the ghost-gluon vertex functions (appendix A.3.2) as follows

$$
\begin{aligned}
\Delta E_{FP} &= \alpha_s \frac{C}{2\sqrt{\Omega_q^{\Sigma}\Omega_{q'}^{\Sigma'}}} 4\pi \sum_{p_1 p_2} \Omega_1 \Omega_2 \left(T_{p_1 q p_2}^{\mathcal{L}\Sigma}\right)\left(T_{p_2 q' p_1}^{\mathcal{L}\Sigma'}\right)^* \\
&\qquad \times i \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} \frac{1}{[\omega_2^2 - \Omega_2^2]} \frac{1}{[(\omega + \omega_2)^2 - \Omega_1^2]}
\end{aligned} \tag{5.23}
$$

In the ghost loop, there are no tadpole contributions which need to be separated off. Thus we may obtain the spectral form for the energy integral immediately from (5.23) in the same fashion as discussed in the previous section

$$
\begin{aligned}
i \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} E(\omega_2) &\equiv i \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} \frac{1}{[\omega_2^2 - \Omega_2^2]} \frac{1}{[(\omega + \omega_2)^2 - \Omega_1^2]} \\
&= -\int_0^1 dt \int_0^{\infty} dz \sqrt{\frac{z}{4\pi}} e^{\omega^2 z t(1-t) - \Omega_1^2 z t - \Omega_2^2 z(1-t)}
\end{aligned} \tag{5.24}
$$

This expression is in Minkowski space. Its contour integral is

$$
i \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} E(\omega_2) = \frac{1}{2\Omega_2 [(\omega + \Omega_2)^2 - \Omega_1^2]} + \frac{1}{2\Omega_1 [(\omega - \Omega_1)^2 - \Omega_2^2]} \tag{5.25}
$$

which may serve as a check on the spectral form.

Finally, the singular and analytic continuation factors for this diagram are given by

$$
\mathcal{S}_{FP}^{\Sigma\Sigma'}(z) = \frac{\alpha_s}{2\sqrt{\Omega_q^{\Sigma}\Omega_{q'}^{\Sigma'}}} \frac{C}{4\pi} \left(\frac{1}{6} q^{\Sigma} q^{\Sigma'} + \frac{1}{12} q^2 g^{\Sigma\Sigma'}\right) \frac{e^{-z}}{z} \tag{5.26}
$$

and

$$
\mathcal{C}_{FP}^{\Sigma\Sigma'}(z) = \frac{\alpha_s}{2\sqrt{\Omega_q^{\Sigma}\Omega_{q'}^{\Sigma'}}} \frac{C}{8\pi} \frac{g^{\Sigma\Sigma'}}{z^2} \tag{5.27}
$$

5.3 The Gluon Loop

The cavity expression for the gluon loop diagram is easiest obtained using the Feynman rules in coordinate space. Note however, that in free space the Feynman rules are always used in conjunction with momentum conservation at the vertices, which makes the calculation easier. In the case of the cavity, momentum conservation can no longer be applied, and thus one has to use the Feynman rules expressed as derivatives of all three gluon lines at the vertices (see appendix D for a discussion of the Feynman rules).

We start with the energy shift, obtained from the free space Feynman rules in coordinate space, together with the restored external legs

$$\begin{aligned}
\Delta E_G &= \frac{i}{2}\int_{-\infty}^{\infty} dt_x \int_{-\infty}^{\infty} dt_y \int d\vec{x} \int d\vec{y}\,(gf_{cda})(gf_{dca'}) \\
&\times i\Big\{ (\partial_\tau g_{\sigma\mu} - \partial_\mu g_{\sigma\tau})_{x_1} + (\partial_\mu g_{\sigma\tau} - \partial_\sigma g_{\mu\tau})_{x_2} + (\partial_\sigma g_{\mu\tau} - \partial_\tau g_{\mu\sigma})_{x_3} \Big\} \\
&\times i\Big\{ (\partial_{\sigma'} g_{\nu\tau'} - \partial_\nu g_{\sigma'\tau'})_{y_2} + (\partial_\nu g_{\sigma'\tau'} - \partial_{\tau'} g_{\sigma'\nu})_{y_1} + (\partial_{\tau'} g_{\nu\sigma'} - \partial_{\sigma'} g_{\nu\tau'})_{y_3} \Big\} \\
&\times \sum_{\Sigma_1, p_1} (-ig^{\Sigma_1\Sigma_1}) \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} \frac{a^\sigma_{\Sigma_1 p_1}(\vec{x_1}) a^{\sigma' *}_{\Sigma_1 p_1}(\vec{y_1})}{\omega_1^2 - \Omega_1^2} e^{-i\omega_1(t_x - t_y)} \\
&\times \sum_{\Sigma_2, p_2} (-ig^{\Sigma_2\Sigma_2}) \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} \frac{a^\tau_{\Sigma_2 p_2}(\vec{x_2}) a^{\tau' *}_{\Sigma_2 p_2}(\vec{y_2})}{\omega_2^2 - \Omega_2^2} e^{-i\omega_2(t_x - t_y)} \\
&\times \frac{1}{2\sqrt{\Omega^\Sigma_q \Omega^{\Sigma'}_{q'}}} a^\mu_{\Sigma q}(\vec{x}_3) a^{\nu *}_{\Sigma' q'}(\vec{y}_3) e^{-i\omega t_x + i\omega' t_y}
\end{aligned} \tag{5.28}$$

The subscripts x_1, etc., on the derivatives are a reminder that these derivatives only act on the wave function with that particular coordinate. At the end, one has to put $x_1 = x_2 = x_3 = x$ and $y_1 = y_2 = y_3 = y$ of course, so this labelling of the coordinates is just to ensure that the correct derivative is taken. As always, the energy Ω_1 denotes the energy of the cavity mode characterized by the quantum numbers p_1 and Σ_1. The factor $\frac{1}{2}$ is the symmetry factor for the diagram.

Multiplying out the tensor structure and writing out explicitly the scalar and vector parts of the expression, we arrive at

$$\begin{aligned}
\Delta E_G &= -ig^2 C \sum_{p_1 p_2} \frac{1}{2\sqrt{\Omega^\Sigma_q \Omega^{\Sigma'}_{q'}}} \int_{-\infty}^{\infty} dt_x\, e^{-i(\omega_1+\omega_2+\omega)t_x} \int_{-\infty}^{\infty} dt_y\, e^{i(\omega_1+\omega_2+\omega')t_y} \\
&\times \sum_{\Sigma_1} g^{\Sigma_1\Sigma_1} \sum_{\Sigma_2} g^{\Sigma_2\Sigma_2} \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} \frac{1}{\omega_1^2 - \Omega_1^2} \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} \frac{1}{\omega_2^2 - \Omega_2^2} \\
&\times \int i d\vec{x} \Big\{ (\vec{\nabla} \times \vec{a}_{\Sigma_1 p_1}) \cdot (\vec{a}_{\Sigma_2 p_2} \times \vec{a}_{\Sigma q}) \\
&+ (\vec{\nabla} \times \vec{a}_{\Sigma_2 p_2}) \cdot (\vec{a}_{\Sigma q} \times \vec{a}_{\Sigma_1 p_1}) + (\vec{\nabla} \times \vec{a}_{\Sigma q}) \cdot (\vec{a}_{\Sigma_1 p_1} \times \vec{a}_{\Sigma_2 p_2}) \\
&+ (\vec{\nabla} a^0_{p_1} + \partial_0 \vec{a}_{\Sigma_1 p_1}) \cdot (\vec{a}_{\Sigma_2 p_2} a^0_q - \vec{a}_{\Sigma q} a^0_{p_2}) \\
&+ (\vec{\nabla} a^0_{p_2} + \partial_0 \vec{a}_{\Sigma_2 p_2}) \cdot (\vec{a}_{\Sigma q} a^0_{p_1} - \vec{a}_{\Sigma_1 p_1} a^0_q) \\
&+ (\vec{\nabla} a^0_q + \partial_0 \vec{a}_{\Sigma q}) \cdot (\vec{a}_{\Sigma_1 p_1} a^0_{p_2} - \vec{a}_{\Sigma_2 p_2} a^0_{p_1}) \Big\}
\end{aligned}$$

$$
\begin{aligned}
&\times \int id\vec{y}\Big\{ (\vec{\nabla} \times \vec{a}_{\Sigma_2 p_2}) \cdot (\vec{a}_{\Sigma_1 p_1} \times \vec{a}_{\Sigma' q'}) \\
&+ (\vec{\nabla} \times \vec{a}_{\Sigma_1 p_1}) \cdot (\vec{a}_{\Sigma' q'} \times \vec{a}_{\Sigma_2 p_2}) + (\vec{\nabla} \times \vec{a}_{\Sigma' q'}) \cdot (\vec{a}_{\Sigma_2 p_2} \times \vec{a}_{\Sigma_1 p_1}) \\
&+ (\vec{\nabla} a^0_{p_2} + \partial_0 \vec{a}_{\Sigma_2 p_2}) \cdot (\vec{a}_{\Sigma_1 p_1} a^0_{q'} - \vec{a}_{\Sigma' q'} a^0_{p_1}) \\
&+ (\vec{\nabla} a^0_{p_1} + \partial_0 \vec{a}_{\Sigma_1 p_1}) \cdot (\vec{a}_{\Sigma' q'} a^0_{p_2} - \vec{a}_{\Sigma_2 p_2} a^0_{q'}) \\
&+ (\vec{\nabla} a^0_{q'} + \partial_0 \vec{a}_{\Sigma' q'}) \cdot (\vec{a}_{\Sigma_2 p_2} a^0_{p_1} - \vec{a}_{\Sigma_1 p_1} a^0_{p_2}) \Big\}^* \qquad (5.29)
\end{aligned}
$$

For simplicity, we have omitted here the space coordinate at which the wave functions are to be taken; this is immediately obvious from the integration variable of the integral in which they occur, so no confusion should arise. In this expression, the symbol $\sum_{\Sigma_1}$ means the following: if one encounters a wave function written as $a^0_{p_1}$, then this denotes the scalar mode and no polarization sum must be taken. On the other hand, a wave function written as a vector $\vec{a}_{\Sigma_1 p_1}$ signifies the fact that this wave function denotes a vector polarization, and thus we must sum over the three vector polarizations $\mathcal{L}, \mathcal{M}, \mathcal{E}$. The sum has been written in this somewhat cryptic fashion to demonstrate explicitly the tensor structure.

Note that in the above expression, we cannot perform the time integrations until the time derivatives have been performed. This time derivative pulls down the continuous energy parameter of the corresponding cavity mode. If one now carries out the time integration, this will reduce to delta functions in the continuous energy parameters.

We are interested in the case where the external gluons characterized by the quantum numbers Σ, q and Σ', q' are transversely polarized magnetic or electric modes. Further, the angular momentum algebra restricts the states such that the in-going and out-going quantum numbers should be the same, so we put $q = q'$ and $\Sigma = \Sigma'$ from the outset. This eliminates some of the terms in the expression above. It is also obvious that in the sum over polarizations a product between wave functions of different polarizations, but belonging to the same propagator sum, does not occur, so for example, one does not encounter a combination like $a^0_{p_1} \vec{a}_{\Sigma_1 p_1}$. This further restricts the above sum, so that one ends up with

$$
\begin{aligned}
\Delta E_G \;=\; & -\frac{i}{2}\frac{g^2 C}{2\Omega^\Sigma_q} \sum_{p_1 p_2} \sum_{\Sigma_1} \sum_{\Sigma_2} \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} \frac{1}{\omega_1^2 - \Omega_1^2} \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} \frac{1}{\omega_2^2 - \Omega_2^2} \\
& \times \delta(\omega_1 + \omega_2 + \omega) \delta_{\omega,\omega'} \Big\{ \Big(\int d\vec{x} \Big[(\vec{\nabla} \times \vec{a}_{\Sigma_1 p_1}) \cdot (\vec{a}_{\Sigma_2 p_2} \times \vec{a}_{\Sigma q}) \\
& + (\vec{\nabla} \times \vec{a}_{\Sigma_2 p_2}) \cdot (\vec{a}_{\Sigma q} \times \vec{a}_{\Sigma_1 p_1}) + (\vec{\nabla} \times \vec{a}_{\Sigma q}) \cdot (\vec{a}_{\Sigma_1 p_1} \times \vec{a}_{\Sigma_2 p_2}) \Big] \Big)^2 \\
& - \Big(\int d\vec{x} (\omega - \omega_2) \vec{a}_{\Sigma_2 p_2} \cdot \vec{a}_{\Sigma q} a^0_{p_1} \Big)^2 - \Big(\int d\vec{x} (\omega - \omega_1) \vec{a}_{\Sigma q} \cdot \vec{a}_{\Sigma_1 p_1} a^0_{p_2} \Big)^2 \\
& + \Big(\int d\vec{x} (\vec{\nabla} a^0_{p_2} a^0_{p_1} - \vec{\nabla} a^0_{p_1} a^0_{p_2}) \vec{a}_{\Sigma q} \Big)^2 \Big\} \qquad (5.30)
\end{aligned}
$$

This may be further simplified by expressing it in terms of the three-gluon vertex functions

$$
\begin{aligned}
\Delta E_G &= -\frac{i}{2}\frac{g^2 C}{2\Omega_q^{\Sigma}} \sum_{p_1 p_2} \sum_{\Sigma_1 \Sigma_2} \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} \frac{1}{\omega_1^2 - \Omega_1^2} \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} \frac{1}{\omega_2^2 - \Omega_2^2} \\
&\times \left\{ \left(T_{p_2 q p_1}^{\Sigma_2 \Sigma \Sigma_1} + T_{q p_1 p_2}^{\Sigma \Sigma_1 \Sigma_2} + T_{p_1 p_2 q}^{\Sigma_1 \Sigma_2 \Sigma} \right)^2 - \left((\omega - \omega_2) T_{p_2 q p_1}^{\Sigma_2 \Sigma} \right)^2 \right. \\
&\left. - \left((\omega - \omega_1) T_{p_1 q p_2}^{\Sigma_1 \Sigma} \right)^2 + \left(\Omega_2 T_{p_2 q p_1}^{\mathcal{L}\Sigma} - \Omega_1 T_{p_1 q p_2}^{\mathcal{L}\Sigma} \right)^2 \right\} \\
&\times \delta(\omega_1 + \omega_2 + \omega) \delta_{\omega,\omega'}
\end{aligned}
\tag{5.31}
$$

Here it is understood that in the last term, where the longitudinal polarization occurs explicitly, no polarization sum must be taken as this vector mode occurs as a result of a derivative of the scalar mode.

Before we go ahead with the calculation of the time integrals and arrive at the spectral forms, we wish to obtain a suitable separation into tadpole and non-tadpole contributions of this expression. One needs to exercise extreme care in this procedure and closely follow the corresponding free space calculation. In that case, a separation was obtained by cancelling factors of ℓ^2 or $(\ell + k)^2$ from the numerator and denominator. In free coordinate space, one may express the gluon loop in terms of Feynman rules without momentum conservation (see appendix D), and observe momentum conservation only after deriving the integral expression. Thus, in free coordinate space, the gluon loop may be obtained as

$$
\begin{aligned}
i\Pi_G^{\mu\nu}(x,y) &= -\frac{g^2 C}{2} \int \frac{d^D \ell}{(2\pi)^D} \int \frac{d^D q}{(2\pi)^D} \\
&\times i \Big[(\partial_3 - \partial_2)_\sigma g_{\mu\tau} + (\partial_1 - \partial_3)_\tau g_{\mu\sigma} + (\partial_2 - \partial_1)_\mu g_{\sigma\tau} \Big]_x \\
&\times i \Big[(\partial_3 - \partial_1)_{\tau'} g_{\nu\sigma'} + (\partial_2 - \partial_3)_{\sigma'} g_{\nu\tau'} + (\partial_2 - \partial_3)_\nu g_{\sigma'\tau'} \Big]_y \\
&\times \left(\frac{e^{-i\ell\cdot(x_2 - y_2)}}{\ell^2} g^{\tau\tau'} \right) \left(\frac{e^{-iq\cdot(x_1 - y_1)}}{q^2} g^{\sigma\sigma'} \right)
\end{aligned}
\tag{5.32}
$$

In this expression, tadpole terms proportional to ℓ^2 and q^2 are obtained from second derivatives of the propagator

$$
\begin{aligned}
\partial_{x_2^\sigma} g_{\mu\tau} \partial_{y_2^{\sigma'}} g_{\nu\tau'} D^{\tau\tau'}(x_2, y_2) D^{\sigma\sigma'}(x_1, y_1) &= \ell_\sigma \ell_{\sigma'} \frac{e^{-i\ell\cdot(x_2 - y_2)} g_{\mu\nu}}{\ell^2} \frac{e^{-iq\cdot(x_2 - y_2)} g^{\sigma\sigma'}}{q^2} \\
&= g_{\mu\nu}\, e^{-i\ell\cdot(x_2 - y_2)} \frac{e^{-iq\cdot(x_2 - y_2)}}{q^2}
\end{aligned}
\tag{5.33}
$$

In the cavity, we wish to proceed in direct analogy. One further point to note is, that the four-momentum integral in free space is replaced by a one-dimensional integral over the continuous energy parameter. Correspondingly, we must make the separation only in one dimension instead of four. In other words, we must separate only terms due to the time derivative corresponding to the four-derivative above. We must thus separate the term

$$\frac{(\omega-\omega_1)^2}{(\omega_1^2-\Omega_1^2)(\omega_2^2-\Omega_2^2)} = \frac{\omega^2-2\omega\omega_1+\Omega_1^2}{(\omega_1^2-\Omega_1^2)(\omega_2^2-\Omega_2^2)} + \frac{1}{(\omega_2^2-\Omega_2^2)} \tag{5.34}$$

and similarly the term

$$\frac{(\omega-\omega_2)^2}{(\omega_1^2-\Omega_1^2)(\omega_2^2-\Omega_2^2)} = \frac{\omega^2-2\omega\omega_2+\Omega_2^2}{(\omega_1^2-\Omega_1^2)(\omega_2^2-\Omega_2^2)} + \frac{1}{(\omega_1^2-\Omega_1^2)} \tag{5.35}$$

Now one may proceed to derive the different spectral forms necessary to evaluate (5.31) in its separated form. The time integrations may be performed to yield delta functions, which eliminate one of the ω-integrals. There are five different spectral forms; for the terms without additional factors of ω-terms, the spectral form is just as for the ghost loop:

$$\begin{aligned} i\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}E_0(\omega_2) &\equiv i\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}\frac{1}{\omega_2^2-\Omega_2^2}\int_{-\infty}^{\infty}\frac{d\omega_1}{2\pi}\frac{1}{\omega_1^2-\Omega_1^2}\delta(\omega_1+\omega_2+\omega) \\ &= -\int_0^1 dt\int_0^{\infty}dz\,\sqrt{\frac{z}{4\pi}}e^{z[\omega^2t(1-t)-\Omega_1^2t-\Omega_2^2(1-t)]} \end{aligned} \tag{5.36}$$

For the separation (5.34) we obtain the forms

$$\begin{aligned} i\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}E_1(\omega_2) &\equiv i\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}\frac{\omega^2+2\omega(\omega+\omega_2)+\Omega_1^2}{(\omega_2^2-\Omega_2^2)((\omega+\omega_1)^2-\Omega_1^2)} \\ &= -\int_0^1 dt\int_0^{\infty}dz\sqrt{\frac{z}{4\pi}}\left(\omega^2(1-2t^2)+\Omega_1^2\right)e^{z[\omega^2t(1-t)-\Omega_1^2t-\Omega_2^2(1-t)]} \end{aligned} \tag{5.37}$$

and

$$\begin{aligned} i\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}T_1(\omega_2) &\equiv i\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}\frac{1}{(\omega_2^2-\Omega_2^2)} \\ &= \int_0^{\infty}dz\frac{1}{\sqrt{4\pi z}}e^{-\Omega_2^2zt} \end{aligned} \tag{5.38}$$

whereas for the separation (5.35) one gets instead

$$\begin{aligned} i\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}E_2(\omega_2) &\equiv i\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}\frac{\omega^2-2\omega\omega_2+\Omega_2^2}{(\omega_2^2-\Omega_2^2)((\omega+\omega_1)^2-\Omega_1^2)} \\ &= -\int_0^1 dt\int_0^{\infty}dz\,\sqrt{\frac{z}{4\pi}}\left(\omega^2(1+2t)+\Omega_2^2\right)e^{z[\omega^2t(1-t)-\Omega_1^2t-\Omega_2^2(1-t)]} \end{aligned} \tag{5.39}$$

and

$$
\begin{aligned}
i\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}T_2(\omega_2) &\equiv i\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}\frac{1}{((\omega+\omega_2)_2^2-\Omega_1^2)} \\
&= \int_0^{\infty}dz\frac{1}{\sqrt{4\pi z}}e^{-\Omega_1^2 zt} \qquad (5.40)
\end{aligned}
$$

The singular factor is obtained easily from the free space form

$$
\mathcal{S}_G^{\Sigma\Sigma'}(z)=\frac{\alpha_s}{2\sqrt{\Omega_q^\Sigma\Omega_{q'}^{\Sigma'}}}\frac{C}{12\pi}\left(-\frac{11}{2}q^\Sigma q^{\Sigma'}+\frac{19}{4}q^2 g^{\Sigma\Sigma'}\right)\frac{e^{-z}}{z} \qquad (5.41)
$$

and the analytic continuation for the unseparated form is

$$
\mathcal{C}_G^{\Sigma\Sigma'}(z)=-9\frac{\alpha_s}{2\sqrt{\Omega_q^\Sigma\Omega_{q'}^{\Sigma'}}}\frac{C}{8\pi}\frac{g^{\Sigma\Sigma'}}{z^2} \qquad (5.42)
$$

whereas for the separated form it is

$$
\mathcal{C}_G^{\Sigma\Sigma'}(z)=-7\frac{\alpha_s}{2\sqrt{\Omega_q^\Sigma\Omega_{q'}^{\Sigma'}}}\frac{C}{8\pi}\frac{g^{\Sigma\Sigma'}}{z^2} \qquad (5.43)
$$

5.4 The Gluon Tadpole

Unlike the other three loops, the gluon tadpole involves a single propagator in the loop. That means that it arises from the first term in the perturbative expansion (2.29), and contains only one factor of the interaction Hamiltonian. It also means that one cannot check the summation over cavity modes with a sum rule, as the infinite sum just gives rise to a divergent expression if not properly regularized. Note that the energy shift for this diagram has the same sign as the Feynman amplitude, as this is a first-order diagram. We derive the tadpole diagram using the Feynman rules, as the symmetric form of the Gell-Mann and Low theorem gives rise to a multitude of contractions which are not very enlightening to perform.

The Feynman amplitude in free coordinate space is given by

$$
\begin{aligned}
i\Pi_T^{\mu\nu} &= -\frac{i}{2}g^2\Big(f_{cce}f_{a'ae}(g^{\sigma'\nu}g^{\sigma\mu}-g^{\sigma'\mu}g^{\sigma\nu})+f_{ca'e}f_{aec}(g^{\sigma'\mu}g^{\sigma\nu}-g^{\sigma\sigma'}g^{\mu\nu}) \\
&\quad + f_{cae}f_{ca'e}(g^{\sigma\sigma'}g^{\mu\nu}-g^{\sigma'\nu}g^{\mu\sigma})\Big)\,iD_{\sigma\sigma'}^{cc}(x,y)\,\delta(x,y)\,\delta_{\sigma\sigma'} \qquad (5.44)
\end{aligned}
$$

Inserting the gluon propagator and the cavity modes, and transforming into the cavity gives

$$
\begin{aligned}
\Delta E_T &= -\frac{i}{2}\frac{g^2C}{2\sqrt{\Omega_q^\Sigma\Omega_{q'}^{\Sigma'}}}\sum_{\Sigma_1,p_1}g^{\Sigma_1\Sigma_1}\int d\vec{x}\int_{-\infty}^{\infty}\frac{d\omega_1}{2\pi}\frac{1}{\omega_1^2-\Omega_1^2} \\
&\quad\times\Big((a_{\Sigma_1p_1}\cdot a_{\Sigma q})(a_{\Sigma_1p_1}^*\cdot a_{\Sigma'q'}^*)+(a_{\Sigma_1p_1}^*\cdot a_{\Sigma q})(a_{\Sigma_1p_1}\cdot a_{\Sigma'q'}^*) \\
&\quad -2(a_{\Sigma_1p_1}\cdot a_{\Sigma_1p_1}^*)(a_{\Sigma q}\cdot a_{\Sigma'q'}^*)\Big) \qquad (5.45)
\end{aligned}
$$

As always, Ω_1 means the eigenenergy of the cavity mode characterized by p_1 and the corresponding polarization Σ_1. The dot products between wave functions here denotes a four-vector product, so that the sum is over four polarizations. We are interested in the case where the external gluons are transversely polarized, so we consider only the vector part of the external wave functions. Expanding the wave functions explicitly in terms of vector and scalar contributions gives

$$\begin{aligned}\Delta E_T &= -\frac{i}{2}\frac{g^2 C}{2\sqrt{\Omega_q^{\Sigma}\Omega_{q'}^{\Sigma'}}}\sum_{p_1}\int d\vec{x}\int_{-\infty}^{\infty}\frac{d\omega_1}{2\pi}\frac{1}{\omega_1^2-\Omega_1^2}\times \\ &\left\{\sum_{\Sigma_1=\mathcal{L},\mathcal{M},\mathcal{E}}\left(2(\vec{a}_{\Sigma_1 p_1}\cdot\vec{a}^*_{\Sigma_1 p_1})(\vec{a}_{\Sigma q}\cdot\vec{a}^*_{\Sigma' q'})-(\vec{a}_{\Sigma_1 p_1}\cdot\vec{a}_{\Sigma q})(\vec{a}^*_{\Sigma_1 p_1}\cdot\vec{a}^*_{\Sigma' q'})\right.\right. \\ &\left.\left.-(\vec{a}^*_{\Sigma_1 p_1}\cdot\vec{a}_{\Sigma q})(\vec{a}_{\Sigma_1 p_1}\cdot\vec{a}^*_{\Sigma' q'})\right)+2(a^0_{p_1}a^{0*}_{p_1})(\vec{a}_{\Sigma q}\cdot\vec{a}^*_{\Sigma' q'})\right\}\end{aligned}\tag{5.46}$$

The integrals over the cavity modes are expressed as usual in terms of vertex functions. Inserting the four-gluon vertex functions from appendix B.1 one obtains

$$\begin{aligned}\Delta E_T &= -\frac{i}{2}\frac{g^2 C}{2\sqrt{\Omega_q^{\Sigma}\Omega_{q'}^{\Sigma'}}}\sum_{p_1}\int_{-\infty}^{\infty}\frac{d\omega_1}{2\pi}\frac{1}{\omega_1^2-\Omega_1^2}\times \\ &\left\{\sum_{\Sigma_1=\mathcal{L},\mathcal{M},\mathcal{E}}\left(2F^{\Sigma\Sigma'\Sigma_1\Sigma_1}_{qq'p_1p_1}-2F^{\Sigma_1\Sigma\Sigma_1\Sigma'}_{p_1qp_1q'}\right)+2F^{\Sigma\Sigma'}_{qq'p_1p_1}\right\}\end{aligned}\tag{5.47}$$

The spectral form is easily obtained:

$$i\int_{-\infty}^{\infty}\frac{d\omega_1}{2\pi}\frac{1}{\omega_1^2-\Omega_1^2}=\int_0^{\infty}dz\,\frac{1}{\sqrt{4\pi z}}e^{-\Omega_1^2 z}\tag{5.48}$$

The tadpole diagram is, as its name suggests, of purely tadpole nature, so we do not need to make any separation. As we have seen, its singular part is zero (even in the free space formulation), but not the analytic continuation factor which is given by

$$\mathcal{C}_T^{\Sigma\Sigma'}(z)=\frac{\alpha_s}{2\sqrt{\Omega_q^{\Sigma}\Omega_{q'}^{\Sigma'}}}\frac{3C}{4\pi}\frac{g^{\Sigma\Sigma'}}{z^2}\tag{5.49}$$

Chapter 6

Calculation and Results

6.1 Calculation

Since the quark loop diagram in free space represents a transverse and gauge independent quantity by itself, it may be calculated independently from the three gauge loops. It turns out that the calculation of the quark loop in the cavity is much simpler than that of the other three diagrams, although the method is similar. Thus the discussion of the computation for the quark loop may serve as an illustration for subsequent calculations.

6.1.1 The Quark Loop

As mentioned, one may calculate the quark loop diagram in two different ways:

- Perform a direct calculation without separating off the tadpole contribution to the diagram; in this case, we must subtract off an analytic continuation factor. In other words the leading divergence depends on the integration variable z as $1/z^2$ and must be subtracted off to yield a finite result. This procedure makes the result scheme dependent, as one may subtract off additional constants which are not clearly defined. This point has been discussed in section 2.2.

- Separate off the tadpole contribution to the diagram. In this case, there are two terms between which the leading singularity (i.e. the $1/z^2$-divergence) cancels exactly. As the $1/z$-divergence cancels automatically in any on-shell calculation, there is no need for any subtraction in this method. This is obviously preferable to any method which requires subtractions which may introduce constant subtraction factors in addition to the singular contribution.

The quantity we wish to calculate now is

$$\Delta E_Q = \alpha_s \frac{T}{2\Omega_p^\Sigma} 4\pi \sum_{p_1 p_2} \tilde{Q}^{\Sigma p}_{p_2 p_1} Q^{\Sigma p}_{p_1 p_2} \left(i \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} E(\omega_2) \right) \tag{6.1}$$

The vertex functions $Q^{\Sigma p}_{p_1 p_2}$ may be expressed in terms of the radial integrals and angular factors consisting of 3j-symbols, as discussed in appendix A.3.1. Here $\sum_{p_1}$ means a sum over the principal quantum number ν_1 (all integers except zero), as well as the angular momentum quantum numbers j_1 and μ_1. As discussed in appendix A.1.1, j_1 and μ_1 are determined by the Dirac quantum number κ_1, which runs over all integers except zero. Of course, the angular momentum quantum numbers are constrained such that $|j_2 - J| \leq j_1 \leq j_2 + J$, where J is the angular momentum of the external gluon. The spin sum over μ_1 and μ_2 is easily performed. One ends up with

$$\begin{aligned}
\sum_{p_1 p_2} M^{\Sigma p}_{p_1 p_2} &\equiv 4\pi \sum_{p_1 p_2} \tilde{Q}^{\Sigma p}_{p_2 p_1} Q^{\Sigma p}_{p_1 p_2} \\
&= -\sum_{p_1 p_2} (S^{\Sigma p}_{p_1 p_2})^2 (2j_1+1)(2j_2+1) \begin{pmatrix} j_1 & J & j_2 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}^2
\end{aligned} \tag{6.2}$$

As a check on this quantity, one may formulate a sum rule by summing over the quantum numbers of one of the two loop quarks and fixing the quantum numbers of the other. This sum rule is discussed in appendix C.1 and serves as a strong test on the correctness of the vertex sum. The radial integrals are expressed in terms of Bessel functions (see appendix A.3.1) which are generated either by series expansion, or with the help of recursion relations.

In the unseparated form, the energy integral gives rise to the spectral form

$$\begin{aligned}
i\int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} E(\omega_2) &= -\int_0^1 dt \int_0^{\infty} dz \sqrt{\frac{z}{4\pi}}\, e^{z\left(\omega^2 t(1-t) - \epsilon_2^2(1-t) - \epsilon_1^2 t\right)} \\
&\quad \times \left(-\frac{1}{2z} + \omega^2 t(t-1) + \omega\epsilon_2 - \omega t(\epsilon_1+\epsilon_2) + \epsilon_1\epsilon_2 \right)
\end{aligned} \tag{6.3}$$

In the separated form, the energy integral is a sum of two terms. These spectral forms have been given in (5.10) and (5.11).

In this spectral form, one wishes to perform the integration over z as the very last step in the calculation. Thus the integral over t must be performed first; this is done by expressing the integrals in terms of error functions

$$\begin{aligned}
\mathrm{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x dt\, e^{-t^2} \\
\mathrm{erfc}(x) &= \frac{2}{\sqrt{\pi}} \int_x^{\infty} dt\, e^{-t^2} = 1 - \mathrm{erf}(x)
\end{aligned} \tag{6.4}$$

or normalized error functions

$$\begin{aligned}
\mathrm{nerf}(x) &= e^{x^2} \mathrm{erf}(x) \\
\mathrm{nerfc}(x) &= e^{x^2} \mathrm{erfc}(x)
\end{aligned} \tag{6.5}$$

Let us derive the explicit form for the exponential integral

$$\int_0^{\infty} dz\, F(z) = \int_0^1 dt \int_0^{\infty} dz\, e^{z[\omega^2 t(1-t) - \epsilon_1^2 t - \epsilon_2^2(1-t)]} \tag{6.6}$$

First, we note that the integral over z is going to diverge if

$$\omega^2 t(1-t) - \epsilon_1^2 t - \epsilon_2^2(1-t) > 0 \tag{6.7}$$

By optimizing this exponential factor with respect to t we may establish that this happens whenever

$$\omega > |\epsilon_1| + |\epsilon_2| \tag{6.8}$$

This occurs only for isolated cases in the sum. These particular terms may therefore be excluded from the sum and the corresponding z-form evaluated as a contour integral. Recall that the contour integral for the energy integral (6.3) is given by

$$i\int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} E(\omega_2) = \left(\frac{\Theta(\epsilon_2)\Theta(-\epsilon_1)}{\omega + \epsilon_2 - \epsilon_1} - \frac{\Theta(\epsilon_1)\Theta(-\epsilon_2)}{\omega + \epsilon_2 - \epsilon_1} \right) \tag{6.9}$$

independent of whether the integral has been separated or not. Thus for the terms in the sum which satisfy (6.8), one may just use this contour integral form, then add them to the result afterwards. These individual terms are not going to affect the functional behaviour of the spectral form, which results from the sum over all quantum numbers, so that one or two excluded terms will not alter it.

Now we express the integrand in (6.6) in the form

$$F(z) = \int_0^1 dt\ e^{At^2+Bt+C} \tag{6.10}$$

where

$$\begin{aligned} A &= -z\omega^2 \\ B &= z(\omega^2 + \epsilon_1^2 - \epsilon_2^2) \\ C &= -z\epsilon_1^2 \end{aligned} \tag{6.11}$$

Recalling that A is always negative, we may introduce further the notation

$$\begin{aligned} P_1 &= -\frac{B}{2\sqrt{|A|}} \\ P_2 &= \left(1 - \frac{B}{2|A|}\right)\sqrt{|A|} \end{aligned} \tag{6.12}$$

to write this z-form as

$$F(z) = \frac{e^{P_1^2+C}}{\sqrt{|A|}} \int_{P_1}^{P_2} dt\ e^{-t^2} \tag{6.13}$$

This expression may be evaluated as a difference of error functions which are generated through either a series representation or continued fraction expansion, whichever method is more efficient for a particular argument. However, as the difference between similar quantities is difficult to calculate numerically, it is more convenient to use the normalized error functions instead, so

$$F(z) = \frac{1}{2}\sqrt{\frac{\pi}{|A|}}\left(e^{C}\mathrm{nerfc}(P_1) - e^{P_1^2-P_2^2+C}\mathrm{nerfc}(P_2)\right) \tag{6.14}$$

Even so, extreme care must be taken when evaluating these spectral forms, as often one is subtracting very large and comparable quantities from each other. The contour integral of the spectral form serves as a valuable check on the correct evaluation of the z-form.

The spectral form (6.3), in its separated and in the unseparated form, may now be expressed in terms of error functions as discussed. The (infinite) sum over all cavity modes then produces a smooth function of z, which is the parametric representation of the energy shift ΔE_Q. In the separated form, of course, the singular (i.e. $1/z^2$) behaviour of the two separate spectral forms cancels exactly to yield a finite and integrable function.

Furthermore, often the regularized z-form is proportional to $1/\sqrt{z}$, so that it has an integrable singularity. It is therefore useful to perform a transformation of variables form z to y, where $z = y^2$, in other words

$$\int dz\; F(z) = \int dy\; 2yF(y) \equiv \int dy\; F'(y) \tag{6.15}$$

As discussed before, the sum in (6.2) is truncated at a fixed value of the cavity energy, in other words we let all the quantum numbers run over all permissible values such that the corresponding eigenenergy is below a maximum energy E_{max}. The error introduced by this truncation is small as the terms containing large energies are suppressed by way of the exponential factor, which contains the energies squared. Only if the value of y approaches zero closely enough does the error become appreciable, which happens at a value y_{min} of approximately

$$y_{\mathrm{min}} \simeq \frac{\pi}{E_{\mathrm{max}}} \tag{6.16}$$

For $y < y_{\mathrm{min}}$, the error in the y-form increases rapidly. That means that close to the origin, one does not have access to reliable values of the y-form. The piece of the function in this range must therefore be guessed by extrapolating from the region where it is well-known to the small-y range. The error introduced by this procedure is larger than any of the numerical errors resulting from the evaluation of the y-form; nevertheless, in the case of the quark loop, the error due to the functional extrapolation shows only in the fourth significant figure, as the y-form approaches zero at the origin and the contribution of the missing piece to the integral is therefore small.

6.1.2 The Gauge Loops

The three gauge loops must be evaluated together, since the separate results are not meaningful. It turns out that the calculation of the gauge loops in the subtraction formalism does not work. In other words, when one formulates the gauge loops in terms of unseparated spectral forms and subtracts off the leading order $1/z^2$ divergence, one is left with a remaining divergence proportional to $1/z^{3/2}$. This is a divergence due to the boundary of the cavity, as in free space one encounters only divergent contributions which depend on the integration variable z in even powers, i.e. $1/z$ and $1/z^2$ divergences. On the other hand, the highest divergence that can occur due to the presence of the boundary is half an order lower than the corresponding free space divergence (see chapter 4), so to a free space $1/z^2$ divergence corresponds a $1/z^{3/2}$ divergence due to the boundary. It becomes obvious, then, that when one *subtracts* something from the cavity expression, it spoils the conspiracy between singular boundary contributions which cause them to cancel amongst each other. We are rescued by the separation formulation, for in this method, one does not have to perform any subtractions to arrive at a finite result; the boundary cancellation is retained and one does indeed obtain a finite result from the cavity loops. We therefore discuss in the following the calculation of the gauge loops in the separated form.

The cavity expression for the ghost loop is

$$\begin{aligned}\Delta_{FP} &= -\frac{\alpha_s C}{2\Omega_q^\Sigma}4\pi\sum_{p_1p_2}\Omega^0_{p_1}\Omega^0_{p_2}\left(T^{\mathcal{L}\Sigma}_{p_1qp_2}\right)\left(T^{\mathcal{L}\Sigma'}_{p_2q'p_1}\right)^* \\ &\quad\times\int_0^1 dt\int_0^\infty dz\sqrt{\frac{z}{4\pi}}e^{\omega^2zt(1-t)-\Omega_1^2zt-\Omega_2^2z(1-t)} \qquad (6.17)\end{aligned}$$

The vertex sum

$$\sum_{p_1p_2}N^{\Sigma q}_{p_1p_2} = 4\pi\sum_{p_1p_2}\left(T^{\mathcal{L}\Sigma}_{p_1qp_2}\right)\left(T^{\mathcal{L}\Sigma'}_{p_2q'p_1}\right)^* \qquad (6.18)$$

may be simplified by expanding it in terms of vector harmonics and performing the spin sums. The result is

$$\begin{aligned}\sum_{M_1M_2}N^{\Sigma q}_{p_1p_2} &= (-1)^{J_1+J_2+J}(2J_1+1)(2J_2+1)\sum_{LL'L_1L_2}\alpha^\Sigma_{JL}\alpha^\Sigma_{JL'}\alpha^{\mathcal{L}}_{J_1L_1}\alpha^{\mathcal{L}}_{J_2L_2}\hat{L}_1\hat{L}_2\hat{L}\hat{L}' \\ &\quad\times R_{LL_1J_2}(\Omega^\Sigma_q,\Omega^0_{p_1},\Omega^0_{p_2})R_{L'L_2J_1}(\Omega^\Sigma_q,\Omega^0_{p_2},\Omega^0_{p_1})\begin{pmatrix}L_1 & J_2 & L\\ 0 & 0 & 0\end{pmatrix} \\ &\quad\times\begin{pmatrix}L_2 & J_1 & L'\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix}J & J_2 & J_1\\ L_1 & 1 & L\end{Bmatrix}\begin{Bmatrix}J & J_1 & J_2\\ L_2 & 1 & L'\end{Bmatrix} \qquad (6.19)\end{aligned}$$

where the radial functions have been defined in appendix A.3.2. The vertex integrals may again be checked with the help of a sum rule, which is formulated in appendix C.2. The spectral form is easily expressed in terms of normalized error functions.

Next, the tadpole diagram is given by

$$\begin{aligned}\Delta E_T &= -\frac{1}{2}\frac{\alpha_s C}{2\Omega_q^\Sigma}\sum_{p_1}\int_0^\infty dz\frac{1}{\sqrt{4\pi z}}e^{-\Omega_1^2 z} \\ &\quad\times 4\pi\left\{\sum_{\Sigma_1=\mathcal{L},\mathcal{M},\mathcal{E}}\left(2F^{\Sigma\Sigma'\Sigma_1\Sigma_1}_{qq'p_1p_1}-2F^{\Sigma_1\Sigma\Sigma_1\Sigma'}_{p_1qp_1q'}\right)+2F^{\Sigma\Sigma'}_{qq'p_1p_1}\right\} \qquad (6.20)\end{aligned}$$

The vertex integrals can be simplified by performing the spin sum. This has been done in appendix A.3.4.

Finally, we must evaluate the gluon loop, which is given in (5.31). This expression may be divided up into three distinct parts as follows

$$\begin{aligned}\Delta E_G &= -\frac{1}{2}\frac{\alpha_s C}{2\Omega_q^\Sigma}4\pi\sum_{p_1p_2}\left(\sum_{\Sigma_1\Sigma_2} C^{\Sigma q}_{\Sigma_1 p_1 \Sigma_2 p_2} i\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}E_0(\omega_2)\right.\\ &\quad + D^{\Sigma q}_{p_1p_2} i\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}E_0(\omega_2)\\ &\quad + \sum_{\Sigma_1} Z^{\Sigma q}_{\Sigma_1 p_1 p_2} i\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}\left(E_1(\omega_2)+T_1(\omega_2)\right)\\ &\quad \left. + \sum_{\Sigma_2}\tilde{Z}^{\Sigma q}_{p_1\Sigma_2 p_2} i\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}\left(E_2(\omega_2)+T_2(\omega_2)\right)\right) \end{aligned} \tag{6.21}$$

Here, we have used abbreviations for the various vertex sums. First, the term which depends only on the curls of the wave functions and is thus a sum over the vector polarizations of both loop gluons is given by

$$\sum_{\Sigma_1\Sigma_2} C^{\Sigma q}_{\Sigma_1 p_1 \Sigma_2 p_2} = \sum_{\substack{\Sigma_1,\Sigma_2\\=\mathcal{L},\mathcal{M},\mathcal{E}}}\left(T^{\Sigma_2\Sigma\Sigma_1}_{p_2 q p_1} + T^{\Sigma\Sigma_1\Sigma_2}_{q p_1 p_2} + T^{\Sigma_1\Sigma_2\Sigma}_{p_1 p_2 q}\right)^2 \tag{6.22}$$

Secondly, there is the term which is basically of the same form as the ghost loop

$$D^{\Sigma q}_{p_1p_2} = \left(\Omega_2^0 T^{\mathcal{L}\Sigma}_{p_2 q p_1} - \Omega_1^0 T^{\mathcal{L}\Sigma}_{p_1 q p_2}\right)^2 \tag{6.23}$$

Finally, there are the two terms in which one of the loop gluons is a scalar and the other one a vector gluon

$$\begin{aligned}\sum_{\Sigma_1} Z^{\Sigma q}_{\Sigma_1 p_1 p_2} &= \sum_{\Sigma_1=\mathcal{L},\mathcal{M},\mathcal{E}}\left(T^{\Sigma_1\Sigma}_{p_1 q p_2}\right)^2\\ \sum_{\Sigma_2}\tilde{Z}^{\Sigma q}_{p_1\Sigma_2 p_2} &= \sum_{\Sigma_2=\mathcal{L},\mathcal{M},\mathcal{E}}\left(T^{\Sigma_2\Sigma}_{p_2 q p_1}\right)^2\end{aligned} \tag{6.24}$$

These last terms, $Z^{\Sigma q}_{\Sigma_1 p_1 p_2}$ and its symmetric counterpart, are the only ones for which the corresponding z-forms have been separated (see the discussion of the separation mechanism in section 5.3). The spin sums are performed easily and the procedure is by now standard. All the vertex integrals reduce to sums of radial integrals after the spin sums have been performed. We do not give the explicit results as these would cover too much space. As usual, a sum rule may be formulated for the different vertex sums. This task is performed in appendix C.3. The z-forms have been given in chapter 5 and are easily expressed in terms of the normalized error functions.

6.2 Results

As described, the dimensionless energy shift due to the gluon self-energy is obtained as the integral of the spectral form. We discuss the quark loop and gauge loops separately.

6.2.1 The Quark Loop

We have mentioned that one may calculate the energy shift due to the quark loop in two ways, namely the so-called subtraction and separation methods. In the subtraction method, the result is given as function of the parametric variable z, which diverges as $1/z^2$. From this we must subtract the $1/z^2$ singularity. The explicit form of this singular or so-called analytic continuation factor is obtained from the free space form. In order to transform it into the cavity, the external gluon legs must be restored and we must integrate over the volume of the cavity.

N	J	Σ	Eigenenergy	Energy Shift
1	1	$\mathcal{M}$	2.7437	0.0570
1	2	$\mathcal{M}$	3.8702	0.0632
1	1	$\mathcal{E}$	4.4934	0.0722
1	3	$\mathcal{M}$	4.9734	0.0680
1	2	$\mathcal{E}$	5.7635	0.0696
1	4	$\mathcal{M}$	6.0619	0.0721
2	1	$\mathcal{M}$	6.1168	0.0740
1	3	$\mathcal{E}$	6.9879	0.0685
2	2	$\mathcal{M}$	7.4431	0.0734
2	1	$\mathcal{E}$	7.7253	0.0759
1	4	$\mathcal{E}$	8.1825	0.0678
2	3	$\mathcal{M}$	8.7218	0.0735
2	2	$\mathcal{E}$	9.0950	0.0742
3	1	$\mathcal{M}$	9.3166	0.0763

Table 6.1: *Dimensionless energy shifts due to the quark loop diagram, given for* $\alpha_s = 1$

As discussed before, it is more convenient to express the spectral forms in terms of the parametric variable y rather than in terms of z, where $y = z^2$. The result for the energy shift is thus given as the difference between the cavity y-form $\Delta E(y)$ and the corresponding analytic continuation factor $\mathcal{C}(y)$

$$\Delta E_{\text{finite}} = \int_0^\infty dy \; (\Delta E(y) - \mathcal{C}(y)) \tag{6.25}$$

In figure 6.1 we show these two divergent spectral forms. It is seen that for small

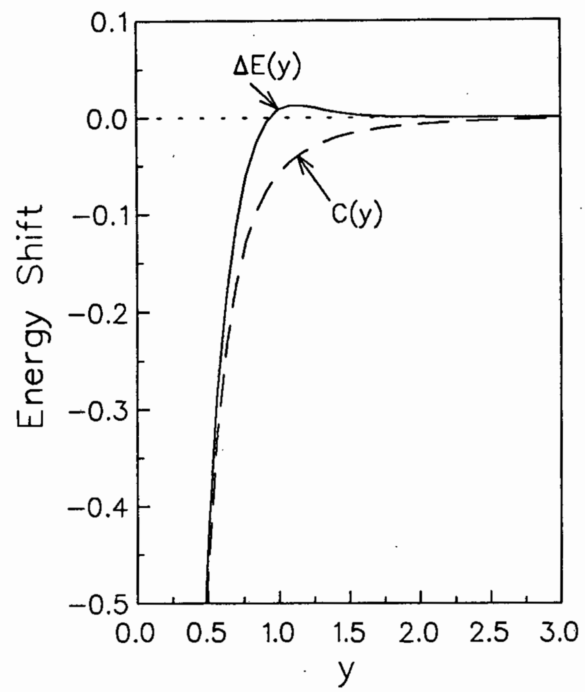


Figure 6.1: *The unseparated quark loop energy shift (solid line), and the free space analytic continuation (dashed line)*

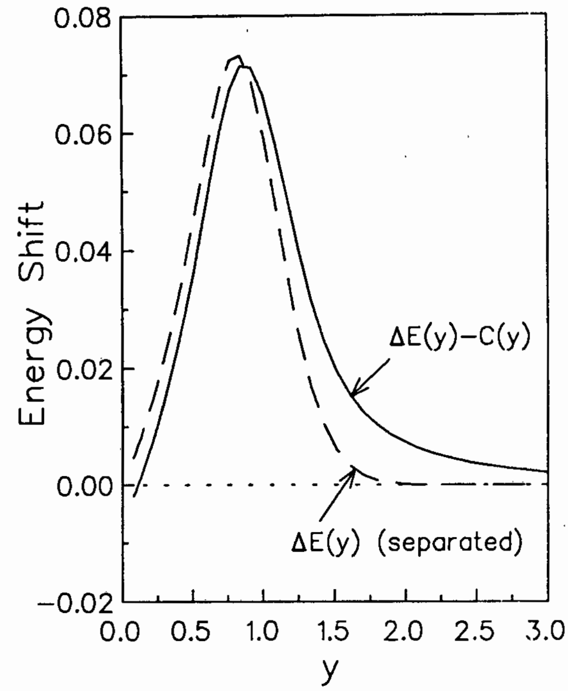


Figure 6.2: *The difference between the unseparated energy shift and the free space analytic continuation (solid line), and the separated energy shift (dashed line), both for the quark loop*

values of y the singular factor obtained from free space falls exactly on the curve obtained from the cavity calculation. In subtracting the two spectral forms, one obtains the regularized spectral form which, when integrated over, leads to the finite contribution to the energy shift. This resulting spectral form is shown in figure 6.2. Note that the spectral forms do not extend all the way to the origin; this is due to the fact that the vertex sum has been truncated at a maximum energy and one does therefore not have access to reliable values for the spectral form below a certain minimum y-value given by (6.16). We stress again that the result obtained by using this method is going to be scheme dependent, and it is not clear what the subtraction constants in this subtraction scheme are. Therefore this method is useful only as a diagnostic check, in order to make sure that the singular behaviour obtained from the cavity calculation is indeed as expected. Also shown in figure 6.2 is the spectral form obtained form the so-called separation method. In this method, the spectral form is separated in such a way that the $1/z^2$ singularity cancels exactly between the two terms obtained in the separation

$$\Delta E_{\text{finite}} = \int_0^\infty dy \ (\Delta E_1(y) + \Delta E_2(y)) \tag{6.26}$$

By comparing the y-forms from the two methods, one can see another merit of the separation method: The y-form in the separation method approaches zero for large values of y. It is therefore easy to integrate over this spectral form, as the error introduced by cutting off the integral at a value of, say, $y = 3$ is negligible. In contrast, the spectral form obtained by ways of the subtraction method has a "tail" that stretches out to very large values of the integration variable. This is due to the nature of the subtraction factor which is not damped exponentially at higher y-values like the cavity expression is.

Not surprisingly, the integrals obtained via the two different methods differ by a small constant (about 0.003 in the dimensionless units). The result obtained from the separation method is scheme independent since it does not depend on any subtraction factor. This is therefore the result we shall quote; it is summarized in table 6.1.

N	J	Σ	Eigenenergy	Energy Shift
1	1	$\mathcal{M}$	2.7437	2.846
1	2	$\mathcal{M}$	3.8702	4.479
1	1	$\mathcal{E}$	4.4934	3.533
1	3	$\mathcal{M}$	4.9734	5.715
1	2	$\mathcal{E}$	5.7635	4.530
1	4	$\mathcal{M}$	6.0619	6.708
2	1	$\mathcal{M}$	6.1168	6.941
1	3	$\mathcal{E}$	6.9879	5.354
2	2	$\mathcal{M}$	7.4431	5.812
2	1	$\mathcal{E}$	7.7253	5.004
1	4	$\mathcal{E}$	8.1825	7.382
2	3	$\mathcal{M}$	8.7218	6.209
2	2	$\mathcal{E}$	9.0950	15.35
3	1	$\mathcal{M}$	9.3166	6.283

Table 6.2: *Dimensionless Energy Shifts due to the gauge loops, for* $\alpha_s = 1$

6.2.2 The Gauge Loops

The calculation of the gauge loop diagrams goes along the same lines as that for the quark loop. In chapter 4, we have given a detailed investigation of the free space expressions for the gauge loops near the boundary and concluded that no new divergences occur due to the presence of the boundary. However, in the investigation we assumed that no subtractions would be necessary to arrive at the result. It turns out that, if one attempts to do the calculation according to the subtraction method described above for the quark loop, there is an additional singularity due to the boundary. As discussed before, we therefore perform the calculation only in the separation formulation.

We have shown in section 5.3 how to arrive at a valid separation for the z-form of the gluon loop. The ghost loop and gluon tadpole diagrams can be calculated directly, as they cannot (and need not) be separated.

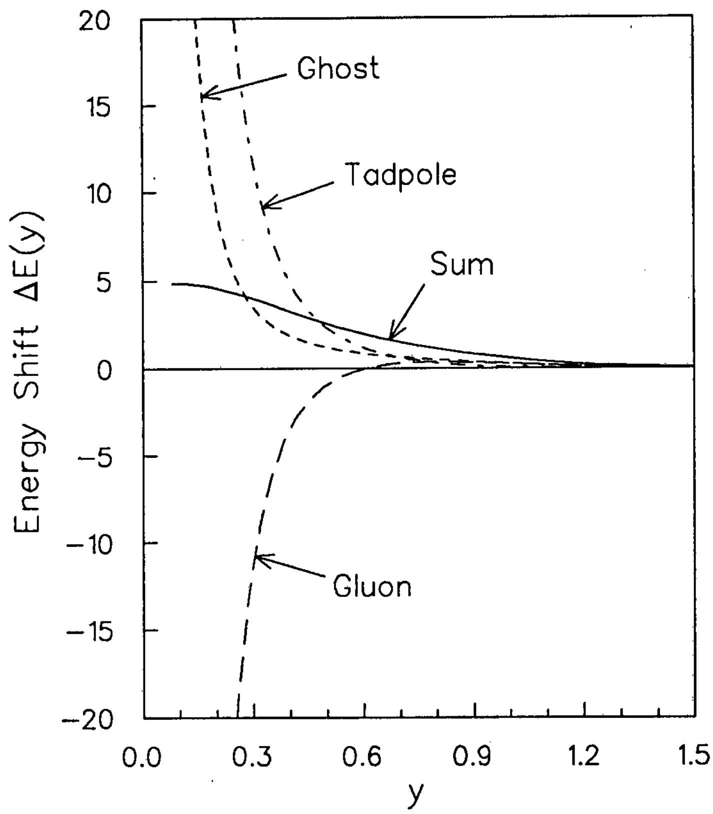


Figure 6.3: *The divergent energy shifts due to the gauge loop diagrams, and the finite sum (solid line) for the lowest lying transverse magnetic gluon*

In figure 6.3 we show the result of this calculation. Plotted are the three divergent y-forms due to the ghost loop, the tadpole and the sum of the two separate parts of the gluon loop. Note that the two separate z-forms of the gluon are to be calculated as separate sums; the two functions are to be added only after the vertex sum has been performed. In this way, the functional dependence on y is such that the divergence cancels exactly between the three terms. This becomes obvious from figure 6.3, where the finite piece is shown as a sum of the three divergent pieces.

The result is the dimensionless energy shift due to the gauge loops, summarized in the table 6.2.

6.2.3 Conclusion

In this thesis, a method has been developed which makes it possible to calculate the gluon self-energy (or other quadratically divergent diagrams) in the cavity without resorting to the subtraction of divergent quantities. In this way, the wave function renormalization, which is necessary in the free space calculation in order to extract from the gluon self-energy a finite and gauge invariant quantity, is made automatic in the cavity computation.

It is important to note that the traditional application of the cavity regularization technique, as introduced in [13], does not work in the case of the gauge loops. In other words, expressing the cavity loop diagram in terms of a spectral function and subtracting from it the free space $1/z^2$ divergence, results in a spectral form with a $1/z^{3/2}$ divergence, which is due to the boundary of the cavity. The way to get around this unfortunate situation is to exploit the fact that several spectral forms can give rise to the same effective divergence, through an analytic continuation of the function in question. Therefore one can perform a separation into spectral forms in such a way that the overall analytic continuation factor between the different forms cancels exactly. This so-called method of separation exists in a similar application for the free space vacuum polarization and is described, for example, by Jauch and Rohrlich [21].

After the calculation has been performed using this method of separation, one is left with a finite contribution to the self-energy which has the effect of shifting the energy spectrum of the gluon cavity modes. The results are shown in tables 6.1 and 6.2. It is seen that the value obtained from the quark loop is almost the same for most of the cavity modes shown, and increases only very slightly with increasing eigen-energy. As expected, the contribution from the gauge loops is much larger than that due to the quark loop. Since the values for the self-energy do not always increase with the eigen-energy, some level-crossing will occur for large values of the strong coupling constant.

The positive gluon self-energy offers a possible explanation for the absence of exotic gluonic states, as its relatively large value decreases the likelihood of their formation. The fact that the contributions from the quark- and gauge loops are both positive is somewhat surprising, as one would expect from the free space vacuum polarization that the values would be of opposite sign.

6.2.4 Acknowledgements

This work was supported in part through a bursary from the Foundation for Research and Development for which I express my sincere thanks.

My supervisor, Prof. Dr. Raoul Viollier, deserves gratitude for his patient support throughout the preparation of this thesis.

I have benefited from many discussions, among others with Dr. Elizabeth Gavin, Prof. Dr. Hank Miller, Dr. Rudolph Tegen and Dr. Gary Tupper. Two people deserve special recognition: I am very grateful to Dr. Andrew Stoddart, who provided many an insight and helped me in the initial stages of this project. A big thank you goes to Michael O'Connor, who was never short of new ideas and provided immeasurable support by checking some of my calculations and keeping the computing facilities in immaculate shape.

Finally, but most importantly, I would like to thank Dr. Detlof von Oertzen for his never failing encouragement and for instilling me with the drive to succeed.

Appendix A

QCD in a Spherical Cavity

In this section we give a brief summary of QCD in a spherical cavity. A detailed account can be found in [4]. The Hamiltonian for QCD can be separated into a free part which does not depend on the coupling constant explicitly,

$$\begin{aligned}\mathcal{H}_0 &= \hat{\bar{\psi}}\left(-\tfrac{1}{2}i\gamma_k \overleftrightarrow{\partial}^k + m\right)\hat{\psi} + \tfrac{1}{4}\left(\partial_k\hat{\boldsymbol{A}}^l - \partial_l\hat{\boldsymbol{A}}^k\right)\cdot\left(\partial_k\hat{\boldsymbol{A}}^l - \partial_l\hat{\boldsymbol{A}}^k\right) + \tfrac{1}{2}\hat{\boldsymbol{\Pi}}^k\cdot\hat{\boldsymbol{\Pi}}^k \\ &\quad - \frac{1}{2\lambda}\hat{\boldsymbol{\Pi}}^0\cdot\hat{\boldsymbol{\Pi}}^0 + \hat{\boldsymbol{\Pi}}^k\cdot\partial_k\hat{\boldsymbol{A}}^0 - \hat{\boldsymbol{\Pi}}^0\cdot\partial_k\hat{\boldsymbol{A}}^k - i\hat{\boldsymbol{\Omega}}\cdot\hat{\boldsymbol{X}} - i\partial_k\hat{\boldsymbol{\chi}}\cdot\partial_k\hat{\boldsymbol{\omega}} \qquad \text{(A.1)}\end{aligned}$$

and an interaction part depending on g

$$\begin{aligned}\mathcal{H}_{int} &= -\tfrac{1}{2}g\hat{\bar{\psi}}\gamma_\mu\boldsymbol{\lambda}\hat{\psi}\cdot\hat{\boldsymbol{A}}^\mu - \tfrac{1}{2}g\left(\partial_k\hat{\boldsymbol{A}}^l - \partial_l\hat{\boldsymbol{A}}^k\right)\cdot\left(\hat{\boldsymbol{A}}^k\times\hat{\boldsymbol{A}}^l\right) \qquad \text{(A.2)} \\ &\quad + g\left(\partial_k\hat{\boldsymbol{A}}^0 + \partial_0\hat{\boldsymbol{A}}^k\right)\cdot\left(\hat{\boldsymbol{A}}^k\times\hat{\boldsymbol{A}}^0\right) - g^2\left(\hat{\boldsymbol{A}}^k\times\hat{\boldsymbol{A}}^0\right)\cdot\left(\hat{\boldsymbol{A}}^k\times\hat{\boldsymbol{A}}^0\right) \\ &\quad + \tfrac{1}{4}g^2\left(\hat{\boldsymbol{A}}^k\times\hat{\boldsymbol{A}}^l\right)\cdot\left(\hat{\boldsymbol{A}}^k\times\hat{\boldsymbol{A}}^l\right) + g\hat{\boldsymbol{\Omega}}\cdot\left(\hat{\boldsymbol{A}}^0\times\hat{\boldsymbol{\omega}}\right) + ig\partial_k\hat{\boldsymbol{\chi}}\cdot\left(\hat{\boldsymbol{A}}^k\times\hat{\boldsymbol{\omega}}\right)\end{aligned}$$

Here, the Hamiltonian is expressed in the usual fashion in terms of the quark field ψ with mass m, the eight gluon fields described by the vector $\boldsymbol{A}^\mu$ and the Faddeev-Popov ghost fields $\boldsymbol{\chi}$ and $\boldsymbol{\omega}$. The corresponding canonically conjugate momenta are given by

$$\begin{aligned}\boldsymbol{\Pi}^k &= \boldsymbol{F}^{k0} \\ \boldsymbol{\Pi}^0 &= -\lambda\partial_\nu\boldsymbol{A}^\nu \\ \boldsymbol{\Omega} &= i\partial_0\boldsymbol{\chi} \qquad \text{(A.3)} \\ \boldsymbol{X} &= -i(\partial_0\boldsymbol{\omega} + g\boldsymbol{A}^0\times\boldsymbol{\omega})\end{aligned}$$

Bold letters are used to denote a vector in the eight-dimensional color space. In order to perform calculations in perturbation theory with the help of the Gell-Mann and Low theorem [20], all quantities must be expressed in the Dirac picture, in which the field operators obey the non-interacting field equations. The Hamiltonian is therefore given here in the Dirac picture, as indicated by the hat on top of the operators.

The fields are confined to a cavity via the MIT boundary conditions, formulated in the Dirac picture as

$$\left(i\, n_k\gamma^k - 1\right)\hat{\psi}\Big|_{\partial V} = \hat{\bar{\psi}}\left(in_k\gamma^k + 1\right)\Big|_{\partial V} = 0 \tag{A.4}$$

$$n_k\left(\partial^k\hat{\boldsymbol{A}}^\nu - \partial^\nu\hat{\boldsymbol{A}}^k\right)\Big|_{\partial V} = n_k\hat{\boldsymbol{A}}^k\Big|_{\partial V} = n_k\partial^k\left(\partial_\nu\hat{\boldsymbol{A}}^\nu\right)\Big|_{\partial V} = 0 \tag{A.5}$$

$$n_k\partial^k\hat{\boldsymbol{\omega}}\Big|_{\partial V} = n_k\partial^k\hat{\boldsymbol{\chi}}\Big|_{\partial V} = 0 \tag{A.6}$$

A.1 The Cavity Modes

A.1.1 The Quark Cavity Modes

The quark wave functions may be expressed in terms of a time-dependent exponential and a time-independent spatial part as follows

$$\psi(q,x) = u_n(\vec{x})e^{-i\omega t} \tag{A.7}$$

The Dirac spinors $u_n(\vec{x})$ are the spherically symmetric solutions to the time-independent Dirac equation, belonging to the energy eigenvalue ϵ_n. They are given by

$$u_n(\vec{r}) = \begin{pmatrix} g_n(r)\chi_\kappa^\mu(\hat{r}) \\ if_n(r)\chi_{-\kappa}^\mu(\hat{r}) \end{pmatrix} \tag{A.8}$$

The index q stands for the collection of quantum numbers $q \equiv \{\omega, n\}$, where the cavity quantum numbers are summarized in the index n

$$n \equiv \{\nu, \kappa, \mu\} \tag{A.9}$$

with ν being the radial, κ the Dirac and μ the magnetic quantum number. The radial quantum number ν can take positive or negative values corresponding to positive and negative energies, and the eigenenergies satisfy the symmetry relation $\epsilon_{\nu,\kappa} = -\epsilon_{-\nu,-\kappa}$. The radial functions $g(r)$ and $f(r)$ are given in terms of spherical Bessel functions as

$$\begin{aligned} g_n(r) &= \frac{\mathcal{N}_n}{R^{3/2}}\, j_l\,(p_n\, r) \\ f_n(r) &= \frac{\mathcal{N}_n p_n \operatorname{sgn}\kappa}{R^{3/2}(\epsilon_n + m)}\, j_{\bar{l}}\,(p_n\, r) \end{aligned} \tag{A.10}$$

where R is the radius of the cavity, p_n the momentum of the cavity mode and the total and angular momentum quantum numbers are defined in the well-known way as a function of the Dirac quantum number κ

$$\begin{aligned} j &= |\kappa| - \tfrac{1}{2} \\ l &= j + \tfrac{1}{2}\mathrm{sgn}\,\kappa \\ \bar{l} &= j - \tfrac{1}{2}\mathrm{sgn}\,\kappa \end{aligned} \tag{A.11}$$

All calculations are performed in terms of dimensionless parameters, so instead of energy, momentum and mass we use

$$\begin{aligned} \zeta &= mR \\ x_n &= p_n R \\ \omega_n &= \epsilon_n R = \mathrm{sgn}\,\nu \sqrt[+]{(x_n^2 + \zeta^2)} \end{aligned} \tag{A.12}$$

The quark momenta are determined by the linear MIT boundary condition (A.4), and the normalization constant is given by

$$(\mathcal{N}_n)^2 = \frac{1}{2\omega_n(\omega_n + \kappa) + \zeta} \left(\frac{x_n}{j_l(x_n)} \right)^2 \tag{A.13}$$

Finally, the quark modes form a complete and orthonormal set of solutions to the Dirac equation, with the orthonormality condition given by

$$\int d\vec{r}\, u_n^\dagger(\vec{r}) u_{n'}(\vec{r}) = \delta_{nn'} \tag{A.14}$$

and the completeness by

$$\sum_n u_{n\alpha}^*(\vec{r}) u_{n\beta}(\vec{r}\,') = \delta_{\alpha\beta}\delta^{(3)}(\vec{r} - \vec{r}\,') \tag{A.15}$$

A.1.2 The Gluon and Ghost Cavity Modes

The gluon field may be expressed in terms of a time-dependent exponential and a time-independent wave function as

$$A_\Sigma^\mu(q, x) = \frac{1}{\sqrt{2\Omega_m^\Sigma}} a_{\Sigma m}^\mu(\vec{x}) e^{-i\omega t} \tag{A.16}$$

where the index q stands for $q \equiv \{\omega, m\}$. The gluon cavity modes obey the time independent d'Alembert equation

$$\begin{aligned} \left(\nabla^2 + (\Omega_m^0)^2\right) a_m^0(\vec{r}) &= 0 \\ \left(\nabla^2 + (\Omega_m^\Sigma)^2\right) \vec{a}_{\Sigma m}(\vec{r}) &= 0 \qquad \Sigma = \mathcal{L}, \mathcal{M}, \mathcal{E} \end{aligned} \tag{A.17}$$

Here Ω_m^Σ is the eigenenergy of the gluon or ghost, and is often abbreviated Ω_m if no confusion can arise as to what the polarization is. The index m stands for the set of quantum numbers $m \equiv \{N, J, M\}$. The spherically symmetric solutions to the equations of motion are given by

$$a_m^0(\vec{r}) = \frac{\mathcal{N}_m^0}{R^{3/2}} i\, j_J(\Omega_m^0 r) Y_{JM}(\hat{r}) \qquad J \geq 0 \tag{A.18}$$

for the ghost and the scalar gluon modes and

$$\begin{aligned}
\vec{a}_{\mathcal{L}m}(\vec{r}) &= \frac{\mathcal{N}_m^{\mathcal{L}}}{R^{3/2}} \frac{1}{\Omega_m^0} \vec{\nabla} j_J(\Omega_m^0 r) Y_{JM}(\hat{r}) \qquad J \geq 0 \\
\vec{a}_{\mathcal{M}m}(\vec{r}) &= \frac{\mathcal{N}_m^{\mathcal{M}}}{R^{3/2}} \frac{\vec{L}}{\sqrt{J(J+1)}} j_J(\Omega_m^{\mathcal{M}} r) Y_{JM}(\hat{r}) \qquad J \geq 1 \\
\vec{a}_{\mathcal{E}m}(\vec{r}) &= \frac{\mathcal{N}_m^{\mathcal{E}}}{R^{3/2}} \frac{1}{i\Omega_m^{\mathcal{E}}} \vec{\nabla} \times \frac{\vec{L}}{\sqrt{J(J+1)}} j_J(\Omega_m^{\mathcal{E}} r) Y_{JM}(\hat{r}) \qquad J \geq 1
\end{aligned} \tag{A.19}$$

for the three vector gluon modes in the longitudinal and transverse magnetic and electric polarizations. Again, the energy eigenvalues are determined by the MIT boundary conditions on the vector fields (A.5) and (A.6). The ghost cavity modes are identical to the scalar gluon modes. The normalization constants are given by

$$\begin{aligned}
(\mathcal{N}_m^0)^2 &= \frac{2}{j_J^2(\Omega_m^0 R)} \left(1 - \frac{J(J+1)}{(\Omega_m^0 R)^2}\right)^{-1} = (\mathcal{N}_m^{\mathcal{L}})^2 \\
(\mathcal{N}_m^{\mathcal{M}})^2 &= \frac{2}{j_J^2(\Omega_m^{\mathcal{M}} R)} \left(1 - \frac{J(J+1)}{(\Omega_m^{\mathcal{M}} R)^2}\right)^{-1} \\
(\mathcal{N}_m^{\mathcal{E}})^2 &= \frac{2}{j_{J+1}^2(\Omega_m^{\mathcal{E}} R)}
\end{aligned} \tag{A.20}$$

The vector gluon modes may be expanded in terms of vector spherical harmonics as follows

$$\vec{a}_{\Sigma m}(\vec{r}) = \frac{\mathcal{N}_m^\Sigma}{R^{3/2}} \sum_{L=|J-1|}^{L=J+1} \alpha_{JL}^\Sigma\, j_L(\Omega_m^\Sigma r) \vec{Y}_{JM}^L(\hat{r}) \tag{A.21}$$

where the non-zero expansion coefficients are given by

$$\alpha_{J,J+1}^{\mathcal{L}} = \sqrt{\frac{J+1}{2J+1}} \qquad \alpha_{J,J-1}^{\mathcal{L}} = \sqrt{\frac{J}{2J+1}} \tag{A.22}$$

$$\alpha_{J,J}^{\mathcal{M}} = 1 \tag{A.23}$$

$$\alpha_{J,J+1}^{\mathcal{E}} = -\sqrt{\frac{J}{2J+1}} \qquad \alpha_{J,J-1}^{\mathcal{E}} = \sqrt{\frac{J+1}{2J+1}} \tag{A.24}$$

Similarly, the curl of the vector modes may be expanded as follows

$$\vec{\nabla} \times \vec{a}_{\Sigma m}(\vec{r}) = -i \frac{\Omega_m^\Sigma \mathcal{N}_m^\Sigma}{R^{3/2}} \sum_{L=|J-1|}^{L=J+1} \beta_{JL}^\Sigma j_L(\Omega_m^\Sigma r) \vec{Y}_{JM}^L(\hat{r}) \tag{A.25}$$

with the non-zero expansion coefficients

$$\beta_{J,J+1}^{\mathcal{M}} = \sqrt{\frac{J}{2J+1}} \qquad \beta_{J,J-1}^{\mathcal{M}} = -\sqrt{\frac{J+1}{2J+1}} \tag{A.26}$$

$$\beta_{J,J}^{\mathcal{E}} = 1 \tag{A.27}$$

The vector cavity modes form a complete and orthonormal set of solutions, where the orthonormality for the gluon modes is

$$\int d\vec{r}\, a_{\Sigma m}^\mu(\vec{r}) a_{\mu\Sigma' m'}^*(\vec{r}) = g^{\Sigma\Sigma'} \delta_{mm'} \tag{A.28}$$

and the completeness relation is

$$\sum_{\Sigma m} g^{\Sigma\Sigma} a_{\Sigma m}^{\mu *}(\vec{r}) a_{\Sigma m}^\nu(\vec{r}\,') = g^{\mu\nu} \delta^{(3)}(\vec{r} - \vec{r}\,') \tag{A.29}$$

The gluon modes behave under complex conjugation as

$$a_{\Sigma m}^{\mu *}(\vec{r}) = \eta_\Sigma (-1)^M a_{\Sigma m^*}^\mu \tag{A.30}$$

where the phase η_Σ is defined

$$\eta_\Sigma = \begin{cases} 1 : \Sigma = \mathcal{L}, \mathcal{E} \\ -1 : \Sigma = 0, \mathcal{M} \end{cases} \tag{A.31}$$

the metric in polarization space is

$$g^{\Sigma\Sigma'} = \begin{cases} 1 : \Sigma = \Sigma' = 0 \\ -1 : \Sigma = \Sigma' = \mathcal{L}, \mathcal{M}, \mathcal{E} \\ 0 : \text{otherwise} \end{cases} \tag{A.32}$$

and the set of quantum numbers m^* stands for $\{N, J, -M\}$. Finally, the divergence operator acting on the scalar gluon or ghost cavity mode has the effect of returning the longitudinal cavity mode as follows

$$a_m^0(\vec{r}) = \frac{-i}{\Omega_m^0} \vec{\nabla} \cdot \vec{a}_{\mathcal{L}m}(\vec{r}) \tag{A.33}$$

and vice versa

$$\vec{a}_{\mathcal{L}m}(\vec{r}) = \frac{-i}{\Omega_m^0} \vec{\nabla} a_m^0(\vec{r}) \tag{A.34}$$

A.2 The Propagators

The Feynman propagators are defined as the vacuum expectation values of the time-ordered products of two field operators. They may be evaluated by inserting the field operators expanded as a sum of cavity modes.

A.2.1 The Quark Propagator

The quark propagator is defined as

$$iS_{\alpha\alpha',cc'}(x,y) = \langle\hat{0}|T\left(\hat{\psi}_{c\alpha}(x)\hat{\overline{\psi}}_{c'\alpha'}(y)\right)|\hat{0}\rangle \tag{A.35}$$

By expanding the field operators in quark cavity modes

$$\begin{aligned}\hat{\psi}_c(x) &= \hat{\psi}_c^{(+)}(x) + \hat{\psi}_c^{(-)}(x) \\ &= \sum_{\substack{\kappa\mu \\ \nu>0}} \left(\hat{a}_{cn}u_n(\vec{x})e^{-i\epsilon_n t} + \hat{b}^\dagger_{cn}u_{-n}(\vec{x})e^{i\epsilon_n t}\right)\end{aligned} \tag{A.36}$$

and observing the anti-commutation relations of the quark creation and annihilation operators

$$\{\hat{a}_{cn}, \hat{a}^\dagger_{c'n'}\} = \{\hat{b}_{cn}, \hat{b}^\dagger_{c'n'}\} = \delta_{cc'}\delta_{nn'} \tag{A.37}$$

where the index c refers to the colour index, one obtains for the propagator the expression

$$\begin{aligned}iS_{\alpha\alpha',cc'}(x,y) &= \delta_{cc'}\sum_{\substack{\kappa\mu \\ \nu>0}} \left(u_{n\alpha}(\vec{x})\overline{u}_{n\alpha'}(\vec{y})\Theta(t_x - t_y)\right. \\ &\quad \left. - u_{-n\alpha}(\vec{x})\overline{u}_{-n\alpha'}(\vec{y})\Theta(t_y - t_x)\right) e^{-i\epsilon_n|t_x - t_y|}\end{aligned} \tag{A.38}$$

Since the color index on the propagator is just a Kronecker delta function, it will often be omitted. Alternatively, the last expression may be written as

$$iS_{\alpha\alpha'}(x,y) = i\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\sum_n \frac{u_{n\alpha}(\vec{x})\overline{u}_{n\alpha'}(\vec{y})}{\omega - \epsilon_n \pm i0}e^{-i\omega(t_x - t_y)} \tag{A.39}$$

as may be verified by direct evaluation of the contour integral in (A.39).

A.2.2 The Ghost Propagator

The ghost propagator is defined

$$i\Delta_{cc'}(x,y) = \langle\hat{0}|T\left(\hat{\omega}_c(x)\hat{\chi}_{c'}(y)\right)|\hat{0}\rangle = -\langle\hat{0}|T\left(\hat{\chi}_c(x)\hat{\omega}_{c'}(y)\right)|\hat{0}\rangle \tag{A.40}$$

Expanding the ghost field in terms of ghost cavity modes, one has

$$\begin{aligned}\hat{\omega}_a(x) &= \hat{\omega}_a^{(+)}(x) + \hat{\omega}_a^{(-)}(x) \\ &= \sum_m \frac{1}{\sqrt{2\Omega^0_m}}\left(\hat{d}_{am}a^0_m(\vec{x})e^{-i\Omega^0_m t} + \hat{d}^\dagger_{am}a^{0*}_m(\vec{x})e^{i\Omega^0_m t}\right)\end{aligned} \tag{A.41}$$

and

$$
\begin{aligned}
\hat{\chi}_a(x) &= \hat{\chi}_a^{(+)}(x) + \hat{\chi}_a^{(-)}(x) \\
&= \sum_m \frac{1}{\sqrt{2\Omega_m^0}} \left(\hat{e}_{am} a_m^0(\vec{x}) e^{-i\Omega_m^0 t} + \hat{e}_{am}^\dagger a_m^{0*}(\vec{x}) e^{i\Omega_m^0 t} \right)
\end{aligned}
\tag{A.42}
$$

Using the ghost creation and annihilation operator anti-commutation relations

$$
\{\hat{d}_{am}, \hat{e}_{a'm'}^\dagger\} = -\{\hat{d}_{am}^\dagger, \hat{e}_{a'm'}\} = i\delta_{aa'}\delta_{mm'} \tag{A.43}
$$

one obtains for the propagator

$$
i\Delta_{aa'}(x,y) = i\delta_{aa'} \sum_m \frac{1}{2\Omega_m^0} a_m^0(\vec{x}) a_m^{0*}(\vec{y}) e^{-i\Omega_m^0 |t_x - t_y|} \tag{A.44}
$$

Alternatively, this can be expressed as

$$
i\Delta_{aa'}(x,y) = -\delta_{aa'} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sum_m \frac{a_m^0(\vec{x}) a_m^{0*}(\vec{y})}{\omega^2 - (\Omega_m^0)^2 + i0} e^{-i\omega(t_x - t_y)} \tag{A.45}
$$

This may be verified by writing the step function in (A.44) in it's integral representation.

A.2.3 The Gluon Propagator

We start from the definition of the gluon propagator

$$
iD_{aa'}^{\mu\nu}(x,y) = \langle \hat{0} | T\left(\hat{A}_a^\mu(x) \hat{A}_{a'}^\nu(y) \right) |\hat{0}\rangle \tag{A.46}
$$

Expanding the gluon field in terms of its cavity modes

$$
\begin{aligned}
\hat{A}_a^\mu(\vec{x}) &= \hat{A}_a^{\mu(+)}(\vec{x}) + \hat{A}_a^{\mu(-)}(\vec{x}) \\
&= \sum_{\Sigma,m} \frac{1}{\sqrt{2\Omega_m^\Sigma}} \left(\hat{c}_{am}^\Sigma a_{\Sigma m}^\mu(\vec{x}) e^{-i\Omega_m^\Sigma t} + \hat{c}_{am}^{\Sigma\dagger} a_{\Sigma m}^{\mu *}(\vec{x}) e^{i\Omega_m^\Sigma t} \right)
\end{aligned}
\tag{A.47}
$$

and furthermore noting the gluon commutation relations

$$
[\hat{c}_{am}^\Sigma, \hat{c}_{a'm'}^{\Sigma'\dagger}] = -g^{\Sigma\Sigma'} \delta_{aa'}\delta_{mm'} \tag{A.48}
$$

one arrives at the gluon propagator

$$
iD_{aa'}^{\mu\nu}(x,y) = -\delta_{aa'} \sum_{\Sigma,m} \frac{g^{\Sigma\Sigma}}{2\Omega_m^\Sigma} a_{\Sigma m}^\mu(\vec{x}) a_{\Sigma m}^{\nu *}(\vec{y}) e^{-i\Omega_m^\Sigma |t_x - t_y|} \tag{A.49}
$$

This may be expressed in the alternate form

$$
iD_{aa'}^{\mu\nu}(x,y) = -i\delta_{aa'} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sum_{\Sigma,m} g^{\Sigma\Sigma} \frac{a_{\Sigma m}^\mu(\vec{x}) a_{\Sigma m}^{\nu *}(\vec{y})}{\omega^2 - (\Omega_m^\Sigma)^2 + i0} e^{-i\omega(t_x - t_y)} \tag{A.50}
$$

A.3 The Vertex Functions

A.3.1 The Quark-Gluon Vertex

The two possible quark-gluon vertex integrals are defined as follows:

$$Q^{\Sigma}_{nn'm} = i \int d\vec{x}\, \overline{u}_n(\vec{x})\gamma_\mu u_{n'}(\vec{x}) a^{\mu}_{\Sigma m}(\vec{x}) \tag{A.51}$$

and

$$\tilde{Q}^{\Sigma}_{nn'm} = i \int d\vec{x}\, \overline{u}_n(\vec{x})\, \gamma_\mu u_{n'}(\vec{x}) a^{\mu *}_{\Sigma m}(\vec{x}) = -Q^{\Sigma}_{n'nm} \tag{A.52}$$

The relationship between the two vertex functions follows from the behaviour of the gluon cavity modes under complex conjugation (A.30). The above integrals may easily be expanded into radial and angular parts

$$Q^{\Sigma}_{nn'm} = \frac{R^{\Sigma}_{nn'm}}{R^{3/2}} \int d\Omega\, \chi^{\mu\dagger}_{\kappa}(\hat{r}) Y_{JM}(\hat{r}) \chi^{\mu'}_{\kappa'}(\hat{r}) \tag{A.53}$$

for the scalar, longitudinal and electric polarizations, and

$$Q^{\mathcal{M}}_{nn'm} = \frac{R^{\mathcal{M}}_{nn'm}}{R^{3/2}} \int d\Omega\, \chi^{\mu\dagger}_{\kappa}(\hat{r}) Y_{JM}(\hat{r}) \chi^{\mu'}_{-\kappa'}(\hat{r}) \tag{A.54}$$

for the magnetic polarization. The angular part is given in terms of the spherical spinors and spherical harmonics by

$$\begin{aligned} &\int d\Omega\, \chi^{\mu\dagger}_{\kappa}(\hat{r}) Y_{JM}(\hat{r}) \chi^{\mu'}_{\kappa'}(\hat{r}) \\ &= (-1)^{\mu+1/2} \frac{\hat{j}\hat{J}\hat{j}'}{\sqrt{4\pi}} \begin{pmatrix} j & J & j' \\ -\mu & M & \mu' \end{pmatrix} \begin{pmatrix} j & J & j' \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \frac{(-1)^{l+J+l'}+1}{2} \end{aligned} \tag{A.55}$$

Here, the notation $\hat{j}$ means $\sqrt{2j+1}$. The radial part is written as

$$\begin{aligned} R^{0}_{nn'm} &= -\mathcal{N}^{0}_{m} \int_0^R r^2 dr\, j_J(\Omega^0_m r) S_{nn'}(r) \\ R^{\mathcal{L}}_{nn'm} &= \frac{\epsilon_{n'} - \epsilon_n}{\Omega^0_m} R^0_{nn'm} \\ R^{\mathcal{M}}_{nn'm} &= \frac{\kappa + \kappa'}{\sqrt{J(J+1)}} \mathcal{N}^{\mathcal{M}}_m \int_0^R r^2 dr\, j_J(\Omega^{\mathcal{M}}_m r) T_{nn'}(r) \\ R^{\mathcal{E}}_{nn'm} &= \frac{\mathcal{N}^{\mathcal{E}}_m}{\Omega^{\mathcal{E}}_m \sqrt{J(J+1)}} \int_0^R r dr\, \Big\{ J(J+1) j_J(\Omega^{\mathcal{E}}_m r) U_{nn'}(r) \\ &\qquad + (\kappa - \kappa') \left(J j_J(\Omega^{\mathcal{E}}_m r) - \Omega^{\mathcal{E}}_m r j_{J-1}(\Omega^{\mathcal{E}}_m r) \right) T_{nn'}(r) \Big\} \end{aligned} \tag{A.56}$$

where the radial integral for the longitudinal mode follows from current conservation. The radial functions are expressed in terms of the quark radial functions from appendix A.1.1 as

$$\begin{aligned} S_{nn'} &= g_n g_{n'} + f_n f_{n'} \\ T_{nn'} &= g_n f_{n'} + f_n g_{n'} \\ U_{nn'} &= g_n f_{n'} - f_n g_{n'} \end{aligned} \tag{A.57}$$

Further, the parity selection which appears as part of the angular integral is usually absorbed in the definition of the radial integral in the following manner

$$S^{\Sigma}_{nn'm} = \frac{(-1)^{l+J+l'}\eta_\Sigma g^{\Sigma\Sigma} + 1}{2} R^{\Sigma}_{nn'm} \tag{A.58}$$

where the phase η_Σ and the metric $g^{\Sigma\Sigma}$ have been defined in appendix A.1.2.

A.3.2 The Ghost-Gluon Vertex

There are two types of ghost-gluon integrals. These are defined by

$$T_{mm'm''} = i \int d\vec{x}\, a^0_m(\vec{x}) a^0_{m'}(\vec{x}) a^0_{m''}(\vec{x}) \tag{A.59}$$

and

$$T^{\Sigma\Sigma'}_{mm'm''} = -i \int d\vec{x}\, \vec{a}_{\Sigma m}(\vec{x}) \cdot \vec{a}_{\Sigma' m'}(\vec{x}) a^0_{m''}(\vec{x}) \tag{A.60}$$

Again, the integrals separate into radial and angular parts

$$T_{mm'm''} = \frac{\mathcal{N}^0_m \mathcal{N}^0_{m'} \mathcal{N}^0_{m''}}{R^{9/2}} R_{JJ'J''}(\Omega^0_m, \Omega^0_{m'}, \Omega^0_{m''}) \int d\Omega\, Y_{JM}(\hat{r}) Y_{J'M'}(\hat{r}) Y_{J''M''}(\hat{r}) \tag{A.61}$$

and

$$\begin{aligned} T^{\Sigma\Sigma'}_{mm'm''} &= \frac{\mathcal{N}^\Sigma_m \mathcal{N}^{\Sigma'}_{m'} \mathcal{N}^0_{m''}}{R^{9/2}} \sum_{LL'} \alpha^\Sigma_{JL} \alpha^{\Sigma'}_{J'L'}\, R_{LL'J''}(\Omega^\Sigma_m, \Omega^{\Sigma'}_{m'}, \Omega^0_{m''}) \\ &\quad \times \int d\Omega\, \vec{Y}^L_{JM}(\hat{r}) \cdot \vec{Y}^{L'}_{J'M'}(\hat{r}) Y_{J''M''}(\hat{r}) \end{aligned} \tag{A.62}$$

The radial integral consists just of a product of Bessel functions with the corresponding indices

$$R_{JJ'J''}(\Omega, \Omega', \Omega'') = \int_0^R r^2 dr\, j_J(\Omega r) j_{J'}(\Omega' r) j_{J''}(\Omega'' r) \tag{A.63}$$

The angular integrals may be evaluated in terms of $3j$- and $6j$-symbols

$$\int d\Omega\, Y_{JM}(\hat{r}) Y_{J'M'}(\hat{r}) Y_{J''M''}(\hat{r}) = \frac{\hat{J}\hat{J}'\hat{J}''}{\sqrt{4\pi}} \begin{pmatrix} J & J' & J'' \\ M & M' & M'' \end{pmatrix} \begin{pmatrix} J & J' & J'' \\ 0 & 0 & 0 \end{pmatrix} \tag{A.64}$$

and

$$\begin{aligned} \int d\Omega\, \vec{Y}^L_{JM}(\hat{r}) \cdot \vec{Y}^{L'}_{J'M'}(\hat{r}) Y_{J''M''}(\hat{r}) &= (-1)^{L+J} \frac{\hat{J}\hat{J}'\hat{J}''\hat{L}\hat{L}'}{\sqrt{4\pi}} \begin{pmatrix} J & J' & J'' \\ M & M' & M'' \end{pmatrix} \times \\ &\quad \begin{pmatrix} L & J'' & L' \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} J' & J'' & J \\ L & 1 & L' \end{Bmatrix} \end{aligned} \tag{A.65}$$

A.3.3 The Three-Gluon Vertex

Apart from the type of integral already encountered in the ghost-gluon vertex, there is the additional three-gluon vertex integral

$$T^{\Sigma\Sigma'\Sigma''}_{mm'm''} = \int d\vec{x}\, \left(\vec{\nabla}\times\vec{a}_{\Sigma m}(\vec{x}\,)\right)\cdot\left(\vec{a}_{\Sigma' m'}(\vec{x}\,)\times\vec{a}_{\Sigma'' m''}(\vec{x}\,)\right) \tag{A.66}$$

This separates into a radial and an angular part in the usual way

$$\begin{aligned} T^{\Sigma\Sigma'\Sigma''}_{mm'm''} &= \frac{\Omega^{\Sigma}_{m}\mathcal{N}^{\Sigma}_{m}\mathcal{N}^{\Sigma'}_{m'}\mathcal{N}^{\Sigma''}_{m''}}{R^{11/2}}\sum_{LL'L''}\alpha^{\Sigma}_{JL}\alpha^{\Sigma'}_{J'L'}\alpha^{\Sigma''}_{J''L''}\,R_{LL'L''}(\Omega^{\Sigma}_{m},\Omega^{\Sigma'}_{m'},\Omega^{\Sigma''}_{m''}) \\ &\quad\times(-i)\int d\Omega\,\vec{Y}^{L}_{JM}(\hat{r})\cdot\left(\vec{Y}^{L'}_{J'M'}(\hat{r})\times\vec{Y}^{L''}_{J''M''}(\hat{r})\right) \end{aligned} \tag{A.67}$$

and the angular integral evaluates to

$$\begin{aligned} \int d\Omega\,\vec{Y}^{L}_{JM}(\hat{r})\cdot\left(\vec{Y}^{L}_{J'M'}(\hat{r})\times\vec{Y}^{L''}_{J''M''}(\hat{r})\right) &= i\sqrt{\frac{3}{2\pi}}\hat{J}\hat{J}'\hat{J}''\hat{L}\hat{L}'\hat{L}''\begin{pmatrix} L & L' & L'' \\ 0 & 0 & 0\end{pmatrix}\times \\ &\quad\begin{pmatrix} J & J' & J'' \\ M & M' & M''\end{pmatrix}\begin{Bmatrix} J & J' & J'' \\ 1 & 1 & 1 \\ L & L' & L''\end{Bmatrix} \end{aligned} \tag{A.68}$$

A.3.4 The Four-Gluon Vertex

The four-gluon vertex is given by integrals of the type

$$\int d\vec{x}\;\vec{a}_{\Sigma m}(\vec{x}\,)\cdot\vec{a}_{\Sigma' m'}(\vec{x}\,)\,a^{0}_{m''}(\vec{x}\,)a^{0}_{m'''}(\vec{x}\,)$$

and

$$\int d\vec{x}\;\vec{a}_{\Sigma m}(\vec{x}\,)\cdot\vec{a}_{\Sigma' m'}(\vec{x}\,)\,\vec{a}_{\Sigma'' m''}(\vec{x}\,)\cdot\vec{a}_{\Sigma''' m'''}(\vec{x}\,)$$

The only time we encounter the four-gluon vertex is in the gluon tadpole diagram, where we have additionally a sum over two of the indices in the above expressions. We discuss the three possible cases separately. Firstly, we have the term

$$\sum_{p}F^{\Sigma\Sigma'}_{qq'pp}\equiv\sum_{p}\int d\vec{x}\,\vec{a}_{\Sigma q}(\vec{x}\,)\cdot\vec{a}^{\,*}_{\Sigma' q'}(\vec{x}\,)\,a^{0}_{p}(\vec{x}\,)a^{0*}_{p}(\vec{x}\,) \tag{A.69}$$

Here, Σ, q and Σ', q' denote the external gluon legs and we put $\Sigma=\Sigma'$ and $q=q'$ in the calculation. The quantum numbers are written $q=\{N,J,M\}$ and $p=\{N_p,J_p,M_p\}$. The sum over the internal loop characterized by the quantum numbers p is for the scalar gluon mode only. We may insert the explicit expressions for the cavity modes into (A.69) to obtain

$$\begin{aligned} \sum_{p}4\pi F^{\Sigma\Sigma'}_{qq'pp} &= 4\pi\frac{(\mathcal{N}^{\Sigma}_{q})^{2}(\mathcal{N}^{0}_{p})^{2}}{R^{6}}\sum_{p}\sum_{L=|J-1|}^{J+1}\sum_{L'=|J-1|}^{J+1}\alpha^{\Sigma}_{JL}\alpha^{\Sigma}_{JL'} \\ &\quad\times\int_{0}^{R}r^{2}dr\,j_{L}(\Omega^{\Sigma}_{q}r)j_{L'}(\Omega^{\Sigma}_{q}r)j^{2}_{J_p}(\Omega^{0}_{p}r) \\ &\quad\times\int d\Omega\,\vec{Y}^{L}_{JM}\cdot(\vec{Y}^{L'}_{JM})^{*}(Y_{J_pM_p})^{2} \end{aligned} \tag{A.70}$$

The sum over spins can be performed using the relation

$$\sum_M |Y_{JM}(\hat{r})|^2 = \frac{2J+1}{4\pi} \tag{A.71}$$

Further we use the orthogonality of the vector spherical harmonics and the definition

$$R_{LL'J_pJ_p} = \int_0^R r^2 dr\, j_L(\Omega_q^\Sigma r) j_{L'}(\Omega_q^\Sigma r) j_{J_p}^2(\Omega_p^0 r) \tag{A.72}$$

to obtain the result

$$\sum_{M_p} 4\pi F_{qqpp}^{\Sigma\Sigma} = \frac{(\mathcal{N}_q^\Sigma)^2(\mathcal{N}_p^0)^2}{R^6} \sum_{L=|J-1|}^{J+1} (\alpha_{JL}^\Sigma)^2 (2J_p+1) R_{LLJ_pJ_p} \tag{A.73}$$

Secondly, we encounter the term

$$\sum_{\Sigma_p,p} F_{qq'pp}^{\Sigma\Sigma'\Sigma_p\Sigma_p} \equiv \sum_{\Sigma_p,p} \int d\vec{x}\, \vec{a}_{\Sigma q}(\vec{x}) \cdot \vec{a}^*_{\Sigma' q'}(\vec{x})\, \vec{a}_{\Sigma_p p}(\vec{x}) \cdot \vec{a}^*_{\Sigma_p p}(\vec{x}) \tag{A.74}$$

Again expanding in terms of cavity modes we obtain

$$\begin{aligned} \sum_{\Sigma_p,p} 4\pi F_{qq'pp}^{\Sigma\Sigma'\Sigma_p\Sigma_p} &= \sum_{\Sigma_p,p} 4\pi \frac{\mathcal{N}_q^\Sigma \mathcal{N}_{q'}^{\Sigma'} (\mathcal{N}_p^{\Sigma_p})^2}{R^6} \sum_{L=|J-1|}^{J+1} \sum_{L'=|J'-1|}^{J'+1} \sum_{L_p=|J_p-1|}^{J_p+1} \sum_{L_p'=|J_p-1|}^{J_p+1} \\ &\quad \times \alpha_{JL}^\Sigma \alpha_{J'L'}^{\Sigma'} \alpha_{J_pL_p}^{\Sigma_p} \alpha_{J_pL_p'}^{\Sigma_p} \int d\Omega\, \vec{Y}_{JM}^L \cdot (\vec{Y}_{J'M'}^{L'})^* \, \vec{Y}_{J_pM_p}^{L_p} \cdot (\vec{Y}_{J_pM_p}^{L_p'})^* \\ &\quad \times \int_0^R r^2 dr\, j_L(\Omega_q^\Sigma r) j_{L'}(\Omega_{q'}^{\Sigma'} r) j_{L_p}(\Omega_p^{\Sigma_p} r) j_{L_p'}(\Omega_p^{\Sigma_p} r) \end{aligned} \tag{A.75}$$

Using the spherical harmonics orthonormality relation and putting $q = q'$, $\Sigma = \Sigma'$, one obtains

$$\begin{aligned} \sum_{M_p} 4\pi F_{qqpp}^{\Sigma\Sigma\Sigma_p\Sigma_p} &= \frac{(\mathcal{N}_q^\Sigma)^2 (\mathcal{N}_p^{\Sigma_p})^2}{R^6} \sum_{L=|J-1|}^{J+1} \sum_{L_p=|J_p-1|}^{J_p+1} (\alpha_{JL}^\Sigma)^2 (\alpha_{J_pL_p}^{\Sigma_p})^2 \\ &\quad \times (2J_p+1) R_{LLL_pL_p} \end{aligned} \tag{A.76}$$

Finally, the term

$$\sum_{\Sigma_p,p} F_{pqpq'}^{\Sigma_p\Sigma\Sigma_p\Sigma'} \equiv \sum_{\Sigma_p,p} \int d\vec{x}\, \vec{a}^*_{\Sigma_p p}(\vec{x}) \cdot \vec{a}_{\Sigma q}(\vec{x})\, \vec{a}_{\Sigma_p p}(\vec{x}) \cdot \vec{a}^*_{\Sigma' q'}(\vec{x}) \tag{A.77}$$

must be considered. Expanding, we get for $\Sigma = \Sigma'$ and $q = q'$

$$\begin{aligned} \sum_{\Sigma_p,p} 4\pi F_{pqpq}^{\Sigma_p\Sigma\Sigma_p\Sigma} &= \sum_{\Sigma_p,p} 4\pi \frac{(\mathcal{N}_q^\Sigma)^2 (\mathcal{N}_p^{\Sigma_p})^2}{R^6} \sum_{L,L'=|J-1|}^{J+1} \sum_{L_p=|J_p-1|}^{J_p+1} \sum_{L_p'=|J_p-1|}^{J_p+1} \\ &\quad \times \alpha_{JL}^\Sigma \alpha_{JL'}^{\Sigma} \alpha_{J_pL_p}^{\Sigma_p} \alpha_{J_pL_p'}^{\Sigma_p} \int d\Omega\, (\vec{Y}_{J_pM_p}^{L_p})^* \cdot \vec{Y}_{JM}^L \, \vec{Y}_{J_pM_p}^{L_p'} \cdot (\vec{Y}_{JM}^{L'})^* \\ &\quad \times \int_0^R r^2 dr\, j_L(\Omega_q^\Sigma r) j_{L'}(\Omega_q^\Sigma r) j_{L_p}(\Omega_p^{\Sigma_p} r) j_{L_p'}(\Omega_p^{\Sigma_p} r) \end{aligned} \tag{A.78}$$

The spin sums can be performed – the necessary relations may be found in any book on angular momentum, see for example [22]. The result is not simple and will not be given here.

Appendix B

Conventions and Integrals

B.1 Feynman Integrals

To demonstrate how these integrals are commonly evaluated in dimensional regularization, let us calculate the integral

$$\mathcal{F} = \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{(\ell^2 - m^2 + i0)((\ell + k)^2 - m^2 + i0)} \tag{B.1}$$

explicitly. Rotating to Euclidean space and elevating the denominators into an exponential one obtains

$$\mathcal{F} = i \int \frac{d^D\ell}{(2\pi)^D} \int_0^\infty dt_1 \int_0^\infty dt_2 e^{-(\ell^2+m^2)t_1} e^{-((\ell+k)^2+m^2)t_2} \tag{B.2}$$

Now we shift the momentum integration according to

$$\ell' = \ell - k\frac{t_2}{t_1 + t_2} \tag{B.3}$$

and subsequently re-label $\ell' \equiv \ell$. Furthermore, performing a change of variables from t_1 and t_2 to z and t

$$\begin{aligned} t_1 &= zt \\ t_2 &= z(1-t) \\ dt_1 dt_2 &= z\,dz\,dt \end{aligned} \tag{B.4}$$

leads to the expression

$$\mathcal{F} = i \int \frac{d^D\ell}{(2\pi)^D} \int_0^\infty zdz \int_0^1 dt\; e^{-(\ell^2+m^2)z-k^2 zt(1-t)} \tag{B.5}$$

The momentum integral now is just a Gaussian integral (appendix B.2) and can easily be evaluated. This gives

$$\mathcal{F} = i \int_0^\infty z\,dz \int_0^1 dt \left(\frac{1}{4\pi z}\right)^{D/2} e^{-m^2 z - k^2 zt(1-t)} \tag{B.6}$$

Finally, we may use the definition of the gamma function in its integral representation

$$\int_0^\infty x^n e^{-ax} dx = \frac{\Gamma(n+1)}{a^{n+1}} \tag{B.7}$$

to evaluate $\mathcal{F}$. In Euclidean space one arrives at

$$\begin{aligned} \mathcal{F} &= \frac{i}{(4\pi)^{D/2}} \int_0^1 dt \frac{\Gamma(\varepsilon)}{[m^2 + k^2 t(1-t)]^\varepsilon} && \text{(B.8)} \\ &= \frac{i}{(4\pi)^{D/2}} \left(\frac{1}{\varepsilon} - \gamma - \int_0^1 dt \ln[m^2 + k^2 t(1-t)] \right) && \text{(B.9)} \end{aligned}$$

Here we have used the well-known relations

$$\begin{aligned} a^\varepsilon &\approx 1 + \varepsilon \ln a && \text{(B.10)} \\ n\Gamma(n) &= \Gamma(n+1) && \text{(B.11)} \end{aligned}$$

To evaluate the integral over t consider the general expression

$$I_n = \int_0^1 t^n \ln(u - t(1-t)) \tag{B.12}$$

We only need the first three integrals in this series; they are given by

$$\begin{aligned} I_0 &= -2 + \ln u + w \ln \frac{w+1}{w-1} \\ I_1 &= \frac{I_0}{2} \\ I_2 &= \frac{1}{3}\left(-\frac{13}{6} + 2u + \ln u + (1-u) w \ln \frac{w+1}{w-1} \right) && \text{(B.13)} \end{aligned}$$

where

$$w = \sqrt{1 - 4u} \tag{B.14}$$

We now write down all the Feynman integrals we are going to need. Since we are also interested in the spectral- or integral form of these integrals, this is also supplied, as well as the result in the zero-mass limit. The subscripts M and E refer to Minkowski and Euclidean space, respectively. We use the abbreviation $\alpha \equiv t(1-t)$. Firstly, the tadpole integral is given by

$$\begin{aligned} \mathcal{T}_M &= \int \frac{d^D \ell}{(2\pi)^D} \frac{1}{\ell^2 - m^2} && \text{(B.15)} \\ \mathcal{T}_E &= -i \int_0^\infty dz \left(\frac{1}{4\pi z} \right)^{D/2} e^{-m^2 z} && \text{(B.16)} \\ &= \frac{i}{(4\pi)^{D/2}} m^2 \left(\frac{1}{\varepsilon} - \gamma + 1 - \ln m^2 \right) && \text{(B.17)} \\ &\xrightarrow{m=0} 0 && \text{(B.18)} \end{aligned}$$

This last step is strictly speaking not correct; however, in the dimensional regularization framework this integral can be consistently defined to be zero in the zero-mass limit. See for example the discussion by Capper and Leibbrandt [17].

The scalar Feynman integral involving only factors of momentum squared in the denominator is given as

$$\mathcal{F}_M = \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{(\ell^2 - m^2)((\ell+k)^2 - m^2)} \tag{B.19}$$

$$\mathcal{F}_E = i\int_0^\infty z dz \int_0^1 dt \left(\frac{1}{4\pi z}\right)^{D/2} e^{-m^2 z - k^2 z\alpha} \tag{B.20}$$

$$= \frac{i}{(4\pi)^{D/2}}\left(\frac{1}{\varepsilon} - \gamma - \int_0^1 dt \ln(m^2 + k^2\alpha)\right) \tag{B.21}$$

$$\mathcal{F}_M \overset{m=0}{\longrightarrow} \frac{i}{(4\pi)^2}\left(\frac{1}{\varepsilon} - \gamma + 2 - \ln\left(\frac{-k^2}{4\pi}\right)\right) \tag{B.22}$$

The calculation of Feynman integrals involving Lorentz indices is done in an analogous fashion. Usually one encounters integrals with one Lorentz index, like

$$\mathcal{F}_M^\mu = \int \frac{d^D\ell}{(2\pi)^D} \frac{\ell^\mu}{(\ell^2 - m^2)((\ell+k)^2 - m^2)}$$

$$\mathcal{F}_E^\mu = -i\int_0^\infty z dz \int_0^1 dt \left(\frac{1}{4\pi z}\right)^{D/2} k^\mu\, t\, e^{-m^2 z - k^2 z\alpha} \tag{B.23}$$

$$= -\frac{i}{(4\pi)^{D/2}} k^\mu \frac{1}{2}\left(\frac{1}{\varepsilon} - \gamma - 2\int_0^1 dt\, t\, \ln(m^2 + k^2\alpha)\right) \tag{B.24}$$

$$\mathcal{F}_M^\mu \overset{m=0}{\longrightarrow} -\frac{i}{(4\pi)^2}\frac{k^\mu}{2}\left(\frac{1}{\varepsilon} - \gamma + 2 - \ln\left(\frac{-k^2}{4\pi}\right)\right) \tag{B.25}$$

Sometimes the Feynman integral with two Lorentz indices is needed. It is given by

$$\mathcal{F}_M^{\mu\nu} = \int \frac{d^D\ell}{(2\pi)^D} \frac{\ell^\mu \ell^\nu}{(\ell^2 - m^2)((\ell+k)^2 - m^2)} \tag{B.26}$$

$$\mathcal{F}_E^{\mu\nu} = i\int_0^\infty z dz \int_0^1 dt \left(\frac{1}{4\pi z}\right)^{D/2} \left[k^\mu k^\nu t^2 + \frac{g^{\mu\nu}}{2z}\right] e^{-m^2 z - k^2 z\alpha} \tag{B.27}$$

$$= \frac{i}{(4\pi)^{D/2}}\left\{k^\mu k^\nu \frac{1}{3}\left[\frac{1}{\varepsilon} - \gamma - 3\int_0^1 dt\, t^2 \ln(m^2 + k^2\alpha)\right]\right. \tag{B.28}$$

$$\left. -\tfrac{1}{2} g^{\mu\nu}\left[\left(m^2 + \frac{k^2}{6}\right)\left(\frac{1}{\varepsilon} - \gamma + 1\right) - \int_0^1 (m^2 + k^2\alpha)\ln(m^2 + k^2\alpha)\right]\right\}$$

$$\mathcal{F}_M^{\mu\nu} \overset{m=0}{\longrightarrow} -\frac{i}{(4\pi)^2}\left\{k^\mu k^\nu \frac{1}{3}\left(\frac{1}{\varepsilon} - \gamma + \frac{13}{6} - \ln\left(\frac{-k^2}{4\pi}\right)\right)\right.$$

$$\left. -k^2 g^{\mu\nu}\frac{1}{12}\left(\frac{1}{\varepsilon} - \gamma + \frac{8}{3} - \ln\left(\frac{-k^2}{4\pi}\right)\right)\right\} \tag{B.29}$$

In the Feynman integrals arising from the reflected propagators, the structure changes slightly. Instead of the gamma function, we now encounter a modified Bessel function

$$
\begin{aligned}
I(\beta,\rho,\nu) &\equiv \int_0^\infty dz\, z^{\nu-1} e^{-\frac{\beta}{z}-\rho z} && \text{(B.30)}\\
&= 2\left(\frac{\beta}{\rho}\right)^{\frac{\nu}{2}} K_\nu(2\sqrt{\beta\rho}) && \text{(B.31)}
\end{aligned}
$$

The modified Bessel function K_ν may be expanded as follows [23]

$$
\begin{aligned}
K_n(z) &= \tfrac{1}{2}\sum_{k=0}^{n-1}(-1)^k \frac{(n-k-1)!}{k!\left(\frac{z}{2}\right)^{n-2k}} && \text{(B.32)}\\
&+ (-1)^{n+1}\sum_{k=0}^{\infty}\frac{\left(\frac{z}{2}\right)^{n+2k}}{k!(n+k)!}\left(\ln\frac{z}{2} - \tfrac{1}{2}\psi(k+1) - \tfrac{1}{2}\psi(n+k+1)\right)
\end{aligned}
$$

We now introduce the shorthand

$$
I(\nu) \equiv I(\frac{x^2}{4}, m^2 + k^2\alpha, \nu) \qquad \text{(B.33)}
$$

and assume that $m = 0$. In this case, one arrives at the asymptotic (i.e. leading order for small values of x) behaviour

$$
\begin{aligned}
I(0) &\to -2\ln\frac{xk\sqrt{\alpha}}{2} && \text{(B.34)}\\
I(-1) &\to \frac{4}{x} && \text{(B.35)}\\
I(-2) &\to \left(\frac{4}{x}\right)^2 && \text{(B.36)}
\end{aligned}
$$

We now list the Feynman integrals we need for the reflected propagators, where it is understood that we let the mass vanish at the end of the calculation. The tadpole integrals near the boundary become

$$
\begin{aligned}
\mathcal{T}_{\mathcal{R}M}(x) &= \int \frac{d^D\ell}{(2\pi)^D}\frac{e^{-i\ell x}}{\ell^2 - m^2} && \text{(B.37)}\\
&= -\frac{i}{(4\pi)^{D/2}} I(-1+\varepsilon) && \text{(B.38)}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{T}'_{\mathcal{R}M}(x) &= \int \frac{d^D\ell}{(2\pi)^D}\frac{e^{-i(\ell+k)x}}{\ell^2 - m^2} && \text{(B.39)}\\
&= -\frac{i}{(4\pi)^{D/2}} e^{-ikx} I(-1+\varepsilon) && \text{(B.40)}
\end{aligned}
$$

Similarly, the scalar Feynman integral becomes, in the presence of the boundary,

$$\mathcal{R}_M(x) = \int \frac{d^D\ell}{(2\pi)^D} \frac{e^{-i\ell x}}{(\ell^2-m^2)((\ell+k)^2-m^2)} \tag{B.41}$$

$$\mathcal{R}_E(x) = \frac{i}{(4\pi)^{D/2}} \int_0^1 dt e^{-iktx} I(\varepsilon) \tag{B.42}$$

The vector integral is of the form

$$\mathcal{R}^\mu_M(x) = \int \frac{d^D\ell}{(2\pi)^D} \frac{e^{-i\ell x}\ell^\mu}{(\ell^2-m^2)((\ell+k)^2-m^2)} \tag{B.43}$$

$$\mathcal{R}^\mu_E(x) = \frac{i}{(4\pi)^{D/2}} \int_0^1 dt e^{-iktx} \left[\frac{i}{2} x^\mu I(-1+\varepsilon) - k^\mu\, t\, I(\varepsilon)\right] \tag{B.44}$$

Finally, the tensor-like integral near the boundary becomes

$$\mathcal{R}^{\mu\nu}_M(x) = \int \frac{d^D\ell}{(2\pi)^D} \frac{e^{-i\ell x}\ell^\mu\ell^\nu}{(\ell^2-m^2)((\ell+k)^2-m^2)} \tag{B.45}$$

$$\begin{aligned}\mathcal{R}^{\mu\nu}_E(x) = \frac{i}{(4\pi)^{D/2}} \int_0^1 dt e^{-iktx} \Big[&\tfrac{1}{2} g^{\mu\nu} I(-1+\varepsilon) - \tfrac{1}{4} x^\mu x^\nu I(-2+\varepsilon) \\ &-\tfrac{1}{2} it(k^\mu x^\nu + x^\mu k^\nu) I(-1+\varepsilon) + t^2\, k^\mu k^\nu I(\varepsilon)\Big]\end{aligned} \tag{B.46}$$

More useful formulae may be found in [25].

B.2 Gaussian Integrals

The conventional one-dimensional Gaussian integral is

$$\int_{-\infty}^{\infty} \frac{d\ell}{2\pi} e^{-\ell^2 z} = \frac{1}{\sqrt{4\pi z}} \tag{B.47}$$

Taking the derivative with respect to z, we obtain

$$\int_{-\infty}^{\infty} \frac{d\ell}{2\pi} \ell^2 e^{-\ell^2 z} = \frac{1}{2z} \frac{1}{\sqrt{4\pi z}} \tag{B.48}$$

These integrals may be generalized to D dimensions. This generalization is mathematically rigorous and is shown for example in [26]. We give only the result here:

$$\int \frac{d^D\ell}{(2\pi)^D} e^{-\ell^2 z} = \left(\frac{1}{4\pi z}\right)^{D/2} \tag{B.49}$$

$$\int \frac{d^D\ell}{(2\pi)^D} \ell_\mu e^{-\ell^2 z} = 0 \tag{B.50}$$

$$\int \frac{d^D\ell}{(2\pi)^D} \ell^2 e^{-\ell^2 z} = \frac{D}{2z} \left(\frac{1}{4\pi z}\right)^{D/2} \tag{B.51}$$

$$\int \frac{d^D\ell}{(2\pi)^D} \ell_1^2 e^{-\ell^2 z} = \frac{1}{2z} \left(\frac{1}{4\pi z}\right)^{D/2} \tag{B.52}$$

These integrals refer to Euclidean space, where $\ell^2 = \sum_\mu \ell_\mu \ell_\mu$. The last result is obvious since we have basically a one-dimensional Gaussian integral over $\ell_1^2 e^{-\ell_1^2 z}$ and a $D-1$-dimensional one over $e^{-\ell^2 z}$.

B.3 Wick Rotations and Euclidean Space

In Minkowski space we use the metric $g^{\mu\nu} \equiv \text{diag}\{1,-1,-1,-1\}$, so that

$$\begin{aligned} \ell^\mu_M &= (\ell_0, \vec{\ell}) \\ \ell_{\mu M} &= (\ell_0, -\vec{\ell}) \end{aligned} \tag{B.53}$$

and

$$\ell^2 = \ell_0^2 - \vec{\ell}^2 \tag{B.54}$$

Similarly, of course

$$\begin{aligned} \partial^\mu_M &= (\frac{\partial}{\partial t}, -\vec{\nabla}) \\ \partial_{\mu M} &= (\frac{\partial}{\partial t}, \vec{\nabla}) \end{aligned} \tag{B.55}$$

A Wick rotation is a rotation from Minkowski space to Euclidean space where we use the metric $g_E^{\mu\nu} = \delta^{\mu\nu} = \text{diag}\{1,1,1,1\}$, and

$$\ell^\mu_E = \ell_{\mu E} = (\ell_0, \vec{\ell}) \tag{B.56}$$

where

$$\begin{aligned} \ell_{0E} &= -i\ell_{0M} \\ \vec{\ell}_E &= \vec{\ell}_M \end{aligned} \tag{B.57}$$

Then

$$\ell^2 = \ell_{0M}^2 - \vec{\ell}_M^2 = -\ell_{0E}^2 - \vec{\ell}_E^2 \tag{B.58}$$

So writing

$$\ell_E^2 = \ell_{0E}^2 + \vec{\ell}_E^2 \tag{B.59}$$

and replacing $g^{\mu\nu}$ by $-\delta^{\mu\nu}$ will result in the desired transformation from Minkowski to Euclidean space. Note that the expression $k^2 g^{\mu\nu}$ does not change sign under this transformation, but the individual terms k^2 and $g^{\mu\nu}$ do.

Finally, one has for the integration measure

$$d^4k_E = d^3\vec{k}_E\, dk_{4E} = -id^4k_M \tag{B.60}$$

B.4 Conventions

Throughout this thesis, we work in natural units where $\hbar = c = 1$. The natural unit of length in the cavity is the cavity radius R. It is convenient to scale all quantities in the calculations such that they are dimensionless by dividing or multiplying them with the appropriate power of R. This is equivalent to setting $R = 1$ throughout; at the end of the calculation, the correct units may be restored easily by dimensional analysis.

We use the Dirac matrices satisfying the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I \tag{B.61}$$

where I is the unit matrix. We use the representation

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \tag{B.62}$$

with the Pauli matrices given by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{B.63}$$

For angular momentum conventions, we follow [22].

Appendix C

Sum Rules

In order to obtain a check on the summation over cavity modes, one may exploit the orthogonality and completeness relations of the cavity modes to obtain sum rules. We discuss here some of the many different sum rules that were used to check the summing algorithms. Note that for the tadpole diagram, which consists only of a single sum, there is obviously no sum rule, as this single sum alone produces the divergence. In the other three diagrams, one of the two sums may be evaluated analytically via the completeness relation of the cavity modes. The divergence then occurs only as the second sum is performed.

C.1 The Quark Loop Sum Rule

In the cavity expression for the quark loop, one encounters the vertex sum

$$4\pi \sum_{p_1 p_2} \tilde{Q}^{\Sigma p}_{p_2 p_1} Q^{\Sigma' p'}_{p_1 p_2} = -4\pi \sum_{p_1 p_2} \int d\vec{x}\, \overline{u}_{p_2}(\vec{x})\gamma_\mu u_{p_1}(\vec{x}) a^{\mu *}_{\Sigma p}(\vec{x}) \times \int d\vec{y}\, \overline{u}_{p_1}(\vec{y})\gamma_\nu u_{p_2}(\vec{x}) a^{\nu}_{\Sigma' p'}(\vec{y}) \tag{C.1}$$

We discuss here only the case where the external gluon lines have the same polarization and the same quantum numbers, i.e. $\Sigma = \Sigma'$ and $p = p'$. The notation is $p = \{N, J, M\}$, $p_1 = \{n_1, j_1, \mu_1\}$ etc. Performing the sum over p_1 and exploiting the completeness relation of the quark cavity modes, eq.(A.15), one arrives at

$$4\pi \sum_{p_1} \tilde{Q}^{\Sigma p}_{p_2 p_1} Q^{\Sigma p}_{p_1 p_2} = -4\pi \int d\vec{x}\, u^\dagger_{p_2}(\vec{x})\gamma_0\gamma_\mu\gamma_0\gamma_\nu u_{p_2}(\vec{x}) a^{\mu *}_{\Sigma p}(\vec{x}) a^{\nu}_{\Sigma p}(\vec{x}) \tag{C.2}$$

Further, noting that

$$\gamma_0\gamma_\mu\gamma_0\gamma_\nu = 2g_{\mu 0}\gamma_0\gamma_\nu - \gamma_\mu\gamma_\nu \tag{C.3}$$

and expanding explicitly in terms of scalar and vector parts, one obtains

$$\gamma_0\gamma_\mu\gamma_0\gamma_\nu a^{\mu *}_{\Sigma p} a^{\nu}_{\Sigma p} = (a^0_p \gamma_0)^2 - (\vec{a}_{\Sigma p} \cdot \vec{\gamma})^2 = a^{\mu *}_{\Sigma p} a_{\mu \Sigma p} \tag{C.4}$$

With these simplifications, expression (C.2) may now be evaluated explicitly for a particular value of p_2, thus obtaining a check on the sum over p_1. Recall that

$$u_n^\dagger u_n = g_n^2(\chi_\kappa^\mu)^2 + f_n^2(\chi_{-\kappa}^\mu)^2 \tag{C.5}$$

It is convenient to perform the spin sum over μ_2 as well; writing the quark spin functions in terms of the normalized two-spinors and spherical harmonics

$$\chi_\kappa^\mu = \sum_m \left(\begin{array}{cc|c} l & \frac{1}{2} & j \\ \mu - m & m & \mu \end{array} \right) \chi_m Y_{l,\mu-m} \tag{C.6}$$

and noting the spherical harmonics relation

$$\sum_\mu |Y_{l,\mu-m}|^2 = \frac{2j+1}{4\pi} \tag{C.7}$$

and orthonormality

$$\int d\Omega \, |Y_{lm}(\hat{r})|^2 = 1 \tag{C.8}$$

we finally obtain

$$4\pi \sum_{\mu_2, p_1} \tilde{Q}_{p_2 p_1}^{\Sigma p} Q_{p_1 p_2}^{\Sigma p} = -(2j_2+1) \int_0^R r^2 dr \, \left(g_{p_2}^2(r) + f_{p_2}^2(r)\right) \Phi_{\Sigma p}(r) \tag{C.9}$$

where the radial function $\Phi_{\Sigma p}(r)$ becomes

$$\Phi_{\mathcal{M}p}(r) = \frac{(\mathcal{N}_p^{\mathcal{M}})^2}{R^3} j_J^2(\Omega_p^{\mathcal{M}} r) \tag{C.10}$$

for the transverse magnetic and

$$\Phi_{\mathcal{E}p}(r) = \frac{(\mathcal{N}_p^{\mathcal{E}})^2}{R^3} \left(\frac{J+1}{2J+1} j_{J-1}^2(\Omega_p^{\mathcal{E}} r) + \frac{J}{2J+1} j_{J+1}^2(\Omega_p^{\mathcal{E}} r) \right) \tag{C.11}$$

for the electric polarization.

C.2 The Ghost Loop Sum Rule

In the ghost loop diagram we encounter the sum

$$\begin{aligned} 4\pi \sum_{p_1 p_2} M_{p_1 p_2}^{\Sigma q} &= 4\pi \sum_{p_1 p_2} \Omega_{p_1}^0 \Omega_{p_2}^0 \left(T_{p_1 q p_2}^{\mathcal{L}\Sigma}\right) \left(T_{p_2 q p_1}^{\mathcal{L}\Sigma}\right)^* \\ &= 4\pi \sum_{p_1 p_2} \int d\vec{x} \, \vec{a}_{\Sigma q}(\vec{x}) \cdot \vec{\nabla} a_{p_1}^0(\vec{x}) a_{p_2}^0(\vec{x}) \\ &\quad \times \int d\vec{y} \, \left(\vec{a}_{\Sigma q}(\vec{y}) \cdot \vec{\nabla} a_{p_2}^0(\vec{y}) a_{p_1}^0(\vec{y})\right)^* \end{aligned} \tag{C.12}$$

We discuss here only the case where the external gluon legs have the same polarization and quantum numbers, as indicated in the expression above. The notation is $q = \{N, J, M\}, p_1 = \{N_1, J_1, M_1\}$ etc. Summing over p_1 we obtain

$$\begin{aligned} 4\pi \sum_{p_1} M^{\Sigma q}_{p_1 p_2} &= 4\pi \int d\vec{x}\, \vec{a}_{\Sigma q}(\vec{x}) \cdot \vec{\nabla} a^0_{p_2}(\vec{x}) \left(\vec{a}_{\Sigma q}(\vec{x}) \cdot \vec{\nabla} a^0_{p_2}(\vec{x})\right)^* \\ &= (\Omega^0_{p_2})^2 4\pi \int d\vec{x}\, \vec{a}_{\Sigma q}(\vec{x}) \cdot \vec{a}_{\mathcal{L} p_2}(\vec{x}) \left(\vec{a}_{\Sigma q}(\vec{x}) \cdot \vec{a}_{\mathcal{L} p_2}(\vec{x})\right)^* \quad \text{(C.13)} \end{aligned}$$

The cavity modes may now be expanded in the usual way according to (A.21) and the remaining spin sum be performed. The necessary relations can be found in [22]. Noting the recurrence relation for spherical Bessel functions

$$j_{n-1}(x) + j_{n+1}(x) = \frac{2n+1}{x} j_n(x) \quad \text{(C.14)}$$

one obtains, after some manipulation, for the transverse magnetic mode

$$4\pi \sum_{M_2 p_1} M^{\Sigma q}_{p_1 p_2} = \frac{J_2(J_2+1)(2J_2+1)(\mathcal{N}^{\mathcal{M}}_q)^2(\mathcal{N}^0_{p_2})^2}{2R^6} \int_0^R dr\, j_J^2(\Omega^{\mathcal{M}}_q r) j^2_{J_2}(\Omega^0_{p_2} r) \quad \text{(C.15)}$$

The sum rule for the transverse electric mode may be obtained in a similar fashion; eventually, one ends up with

$$\begin{aligned} 4\pi \sum_{M_2 p_1} M^{\Sigma q}_{p_1 p_2} &= \frac{(\mathcal{N}^{\mathcal{E}}_q)^2(\mathcal{N}^0_{p_2})^2}{R^6} \int_0^R dr \,\Big\{ J_2(J_2+1)(2J_2+1)^2 j^2_{J_2}(\Omega^0_{p_2} r) \\ &\quad \times \left((J+1) j^2_{J-1}(\Omega^{\mathcal{E}}_p r) + J j^2_{J+1}(\Omega^{\mathcal{E}}_p r)\right) \\ &\quad + (\Omega^0_{p_2})^2 \frac{J(J+1)(2J+1)}{(\Omega^{\mathcal{E}}_p)^2} j_J^2(\Omega^{\mathcal{E}}_p r) \Big(J_2(J_2-1) j^2_{J_2-1}(\Omega^0_{p_2} r) \\ &\quad + (J_2+1)(J_2+2) j^2_{J_2+1}(\Omega^0_{p_2} r) \\ &\quad - 6 J_2(J_2+1) j_{J_2-1}(\Omega^0_{p_2} r) j_{J_2+1}(\Omega^0_{p_2} r)\Big) \Big\} \quad \text{(C.16)} \end{aligned}$$

C.3 The Gluon Loop Sum Rules

In the gluon loop, various different vertices occur, so that it is difficult to obtain a sum rule for the entire expression. Instead, we work out sum rules for the individual terms in the gluon loop.

Firstly, consider the part of the gluon vertex that contains both loop gluons in vector polarizations. In this case there is a contribution to the sum of terms like

$$\begin{aligned} \sum_{\Sigma_1 p_1 \Sigma_2 p_2} C^{\Sigma q}_{\Sigma_1 p_1 \Sigma_2 p_2} &= 4\pi \int d\vec{x} \left(\vec{\nabla} \times \vec{a}_{\Sigma q}(\vec{x})\right) \cdot \left(\vec{a}_{\Sigma_1 p_1}(\vec{x}) \times \vec{a}_{\Sigma_2 p_2}(\vec{x})\right) \\ &\quad \times \int d\vec{y} \left(\vec{\nabla} \times \vec{a}_{\Sigma q}(\vec{y})\right)^* \cdot \left(\vec{a}_{\Sigma_1 p_1}(\vec{y}) \times \vec{a}_{\Sigma_2 p_2}(\vec{y})\right)^* \quad \text{(C.17)} \end{aligned}$$

We consider only this particular term in the expansion, as all the other terms similar to this one are calculated in the same way, and a check on one term should be sufficient. We may write (C.17) explicitly in terms of components and, noting the gluon mode completeness relation (A.29), perform the sum over Σ_1 (the scalar mode does not contribute) and p_1 to obtain

$$\sum_{\Sigma_1 p_1} C^{\Sigma q}_{\Sigma_1 p_1 \Sigma_2 p_2} = 4\pi \varepsilon_{ijk}\, \varepsilon_{lmn}\, \delta_{jm} \int d\vec{x}\, \left(\vec{\nabla} \times \vec{a}_{\Sigma q}(\vec{x})\right)^i \left(\vec{\nabla} \times \vec{a}_{\Sigma q}(\vec{x})\right)^{l*} a^k_{p_2}(\vec{x}) a^n_{p_2}(\vec{x}) \tag{C.18}$$

Noting the relation

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \tag{C.19}$$

this reduces to

$$\begin{aligned}\sum_{\Sigma_1 p_1} C^{\Sigma q}_{\Sigma_1 p_1 \Sigma_2 p_2} &= 4\pi \int d\vec{x} \Big[\left(\vec{\nabla} \times \vec{a}_{\Sigma q}(\vec{x})\right) \cdot \left(\vec{\nabla} \times \vec{a}_{\Sigma q}(\vec{x})\right)^* \vec{a}_{p_2}(\vec{x}) \cdot \vec{a}^*_{p_2}(\vec{x}) \\ &\quad - \left(\vec{\nabla} \times \vec{a}_{\Sigma q}(\vec{x})\right) \cdot \vec{a}^*_{p_2}(\vec{x}) \left(\vec{\nabla} \times \vec{a}_{\Sigma q}(\vec{x})\right)^* \cdot \vec{a}_{p_2}(\vec{x}) \Big]\end{aligned} \tag{C.20}$$

As usual, we may now expand this in terms of vector spherical harmonics, then use the spherical harmonics completeness and orthogonality conditions to perform the spin sums and simplify the result. This yields

$$\begin{aligned}\sum_{M_2, \Sigma_1 p_1} C^{\Sigma q}_{\Sigma_1 p_1 \Sigma_2 p_2} &= \frac{(\Omega^\Sigma_q)^2 (\mathcal{N}^\Sigma_q)^2 (\mathcal{N}^{\Sigma_2}_{p_2})^2}{R^6} \Big\{ (2J_2 + 1) \sum_{LL_2} (\beta^\Sigma_{LJ})^2 (\alpha^{\Sigma_2}_{L_2 J_2})^2 \\ &\quad \times \int_0^R r^2 dr\, j^2_L(\Omega^\Sigma_q r) j^2_{L_2}(\Omega^{\Sigma_2}_{p_2} r) \\ &\quad + 4\pi \sum_{LL'L_2L'_2} \beta^\Sigma_{LJ} \beta^\Sigma_{L'J} \alpha^{\Sigma_2}_{L_2 J_2} \alpha^{\Sigma_2}_{L'_2 J_2} \\ &\quad \times \sum_{M_2} \int d\Omega\, \left(\vec{Y}^L_{JM}(\hat{r}) \cdot \vec{Y}^{L_2 *}_{J_2 M_2}(\hat{r})\right) \left(\vec{Y}^{L'*}_{JM}(\hat{r}) \cdot \vec{Y}^{L'_2}_{J_2 M_2}(\hat{r})\right) \\ &\quad \times \int_0^R r^2 dr\, j_L(\Omega^\Sigma_q r) j_{L'}(\Omega^\Sigma_q r) j_{L_2}(\Omega^{\Sigma_2}_{p_2} r) j_{L'_2}(\Omega^{\Sigma_2}_{p_2} r) \Big\}\end{aligned} \tag{C.21}$$

This may be further simplified with the summation relations for the spherical harmonics. We do not give the detailed result here.

Secondly, one encounters terms like

$$\sum_{p_1 \Sigma_2 p_2} T^{\Sigma q}_{p_1 \Sigma_2 p_2} = 4\pi \int d\vec{x}\, \left(\vec{a}_{\Sigma q}(\vec{x}) \cdot \vec{a}_{\Sigma_2 p_2}(\vec{x})\right) a^0_{p_1}(\vec{x}) \int d\vec{y}\, \left(\vec{a}_{\Sigma q}(\vec{y}) \cdot \vec{a}_{\Sigma_2 p_2}(\vec{y})\right)^* a^{0*}_{p_1}(\vec{y}) \tag{C.22}$$

Performing the sum over p_1 yields

$$\sum_{p_1} T^{\Sigma q}_{p_1 \Sigma_2 p_2} = 4\pi \int d\vec{x}\, |\vec{a}_{\Sigma q}(\vec{x}) \cdot \vec{a}_{\Sigma_2 p_2}(\vec{x})|^2 \tag{C.23}$$

Expanding in terms of spherical harmonics, this reduces to

$$\begin{aligned}\sum_{p_1} T^{\Sigma q}_{p_1 \Sigma_2 p_2} &= \sum_{LL'L_2L_2'} \alpha^{\Sigma}_{JL}\alpha^{\Sigma}_{JL'}\alpha^{\Sigma_2}_{J_2L_2}\alpha^{\Sigma_2}_{J_2L_2'} \\ &\quad \times \sum_{M_2} \int d\Omega \, \left(\vec{Y}^{L}_{JM}(\hat{r}) \cdot \vec{Y}^{L_2}_{J_2M_2}(\hat{r})\right) \left(\vec{Y}^{L'}_{JM}(\hat{r}) \cdot \vec{Y}^{L_2'}_{J_2M_2}(\hat{r})\right)^* \\ &\quad \times \int_0^R r^2 dr \, j_L(\Omega^{\Sigma}_q r) j_{L'}(\Omega^{\Sigma}_q r) j_{L_2}(\Omega^{\Sigma_2}_{p_2} r) j_{L_2'}(\Omega^{\Sigma_2}_{p_2} r) \end{aligned} \quad \text{(C.24)}$$

The procedure of simplification is by now standard, the necessary formulae are given in [22].

Finally, a sum rule may be found for terms like

$$\sum_{p_1 p_2} D^{\Sigma q}_{p_1 p_2} = 4\pi \int d\vec{x} \left(\vec{\nabla} a^0_{p_2}(\vec{x}) \cdot \vec{a}_{\Sigma q}(\vec{x})\right) a^0_{p_1}(\vec{x}) \int d\vec{y} \, \left(\vec{\nabla} a^0_{p_2}(\vec{y}) \cdot \vec{a}_{\Sigma q}(\vec{y})\right)^* a^0_{p_1}(\vec{y})^* \quad \text{(C.25)}$$

Performing the sum over p_1 gives

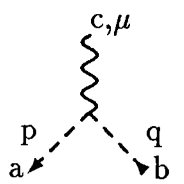
$$\begin{aligned}\sum_{p_1} D^{\Sigma q}_{p_1 p_2} &= 4\pi \int d\vec{x} \, \left(\vec{\nabla} a^0_{p_2}(\vec{x}) \cdot \vec{a}_{\Sigma q}(\vec{x})\right) \left(\vec{\nabla} a^0_{p_2}(\vec{x}) \cdot \vec{a}_{\Sigma q}(\vec{x})\right)^* \\ &= 4\pi (\Omega^0_{p_2})^2 \int d\vec{x} \, \left(\vec{a}^{\mathcal{L}}_{p_2}(\vec{x}) \cdot \vec{a}_{\Sigma q}(\vec{x})\right) \left(\vec{a}^{\mathcal{L}}_{p_2}(\vec{x}) \cdot \vec{a}_{\Sigma q}(\vec{x})\right)^* \end{aligned} \quad \text{(C.26)}$$

In this form, the expression is just a special case of the previous one, (C.23) and may be calculated in the same way. The result is not given here.

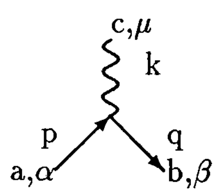
Appendix D

The Feynman Rules

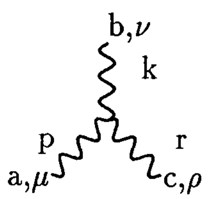
The momentum space Feynman rules in the Lorentz gauge are given by (see for example [24]):



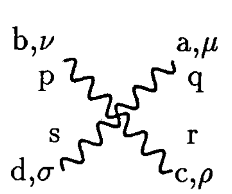
$$-gf^{abc}p_\mu \tag{D.1}$$



$$-ig\gamma_\mu \frac{\lambda^c_{ab}}{2} \tag{D.2}$$



$$-gf^{abc}\{(k-r)_\mu g_{\nu\rho} + (r-p)_\nu g_{\mu\rho} + (p-k)_\rho g_{\mu\nu}\} \tag{D.3}$$

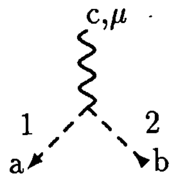


$$\begin{aligned}&-ig^2 f^{abe} f^{cde}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})\\ &-ig^2 f^{ace} f^{bed}(g_{\mu\sigma}g_{\rho\nu} - g_{\mu\nu}g_{\rho\sigma})\\ &-ig^2 f^{ade} f^{bce}(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\rho}g_{\sigma\nu})\end{aligned} \tag{D.4}$$

For the Feynman rules in coordinate space, we replace any factor of p_μ occuring in the momentum space Feynman rules by a differentiation with respect to the x-coordinate of the corresponding propagator. In free coordinate space, one usually formulates the Feynman rules such that four-momentum is conserved at each vertex.

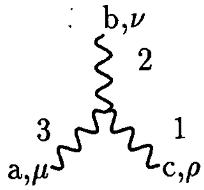
This means that in the case of the three-gluon vertex, the differentiation with respect to one of the legs is replaced by differentiation with respect to the other two legs. In the cavity, four-momentum is not conserved and this is therefore no longer valid. Hence it is here more convenient to leave the coordinate space Feynman rule for the three-gluon vertex, which are going to be needed for the gluon loop diagram, formulated in its totally anti-symmetric form, i.e. in terms of derivatives of all three legs.

The Feynman rule for the ghost-gluon vertex in coordinate space is straightforward: We get



$$-gf^{abc}(-i\partial_1)_\mu \tag{D.5}$$

The three-gluon vertex is given by



$$\begin{aligned} &-gf^{abc}\{(i\partial_1 - i\partial_2)_\mu g_{\nu\rho} \\ &+(i\partial_3 - i\partial_1)_\nu g_{\mu\rho} + (i\partial_2 - i\partial_3)_\rho g_{\mu\nu}\} \end{aligned} \tag{D.6}$$

As discussed previously, we have omitted here a factor μ^ϵ which, when using dimensional regularization, should be multiplied into each factor of the coupling constant g in order to render it dimensionless in arbitrary dimensions. This additional factor is not needed in the cavity, however, since there we are performing all calculations in $D = 4$ dimensions.

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